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UNCERTAINTY EVALUATION OF DYNAMIC MEASUREMENTS IN LINE WITH THE GUM

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Abstract – We consider the task of uncertainty evaluation in the context of dynamic measurements. We assume that the relation between the time-dependent value of the measurand and the available output signal of an employed sensor is governed by a linear time-invariant system. Estimation of the measurand is done by applying a digital filter to the sensor's output signal. We propose a method for uncertainty evaluation which is in line with the *Guide to the Expression of Uncertainty in Measurement* (GUM). The method accounts for dynamic errors due to non-perfect compensation of the dynamic behavior of the sensor, and it allows for real-time calculation when a causal digital filter is employed. The proposed uncertainty calculation method can be used to design an uncertainty-optimum filter. We illustrate the procedures in terms of a simple example.

Keywords: Uncertainty, LTI System, Digital Filter

1. INTRODUCTION

Measurement results need to be accompanied by a statement on their accuracy. This is in particular important in metrology which is concerned with the establishment of measurement units and the realization of measurement standards. In metrology, uncertainty is evaluated according to the *Guide to the Expression of Uncertainty in Measurement* (GUM) [1]. A key feature of the GUM is the 'propagation of uncertainties' assuming a model relation between the measurand and all its influencing quantities. In a recent supplement to the GUM (GUM S1, [2]), the propagation of uncertainties is replaced by the propagation of (degree-of-belief) probability density functions (PDFs). Once the model relation for the measurand has been established and the PDFs associated with all input quantities have been assigned, the calculation of the PDF associated with the measurand is done according to the rules of probability theory; for this, GUM S1 provides an easy-to-use calculation method based on a Monte-Carlo procedure.

The goal of dynamic measurements can be viewed as the task to determine a measurand having a time-dependent value, where the available data from an employed sensor is a time series whose values at a particular time depend on present and *past* values of the measurand. As a consequence, the measurand cannot be determined utilizing only the sensor's static properties. Tools from digital signal processing (DSP) [3] are used for proper analysis, and in particular digital filters are applied for the compensation of the non-perfect dynamic behavior of an employed sensor, cf., e.g., [4-7].

Uncertainty considerations such as the variance of the output signal of a linear time-invariant (LTI) system driven by a noise process have widely been considered in DSP. Unlike the GUM or GUM S1, however, these concepts are usually based on classical statistics. Uncertainty is then characterized in terms of an estimate of the sampling variance of the considered estimator of the measurand, which may be accompanied by an upper bound of the systematic error (in this context called dynamic error). The GUM, on the other hand, provides a consistent treatment of uncertainties stemming from random and systematic errors. The main task of uncertainty evaluation for dynamic measurements thus seems to align the techniques well-known in DSP with the framework of uncertainty evaluation in metrology. While the current guidelines [1,2] give no explicit guidance for uncertainty evaluation in dynamic measurements, their methodology can be adopted to this case when DSP is considered and this paper gives such an attempt.

After having stated the generic task to be considered, we show that the current guidelines for uncertainty evaluation are readily applicable. When the dynamic error is not negligible, it needs to be taken into account. To do so, we give a bound on the dynamic error. Utilizing this bound, the uncertainty due to a remaining dynamic error can then be accounted for in a straightforward way according to the GUM. When a causal digital filter is applied, calculations may be carried out in real-time. The resulting combined uncertainty also allows for the design of an uncertainty-optimum filter. We finally illustrate the proposed procedures in terms of a simple example.

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2. PROBLEM SPECIFICATION AND ASSUMPTIONS

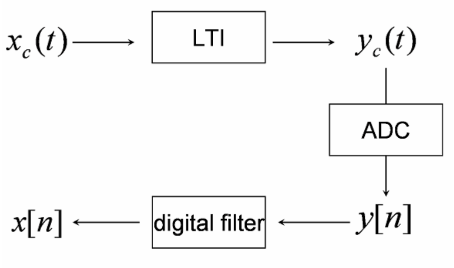

Fig. 1. Illustration of the considered task of analysis.

The time-dependent value $x_c(t)$ of a measurand acts as the input to a sensor and it is to be determined from the sensor's output signal $y[n] = y_c(n \cdot T_s)$, for which an estimate is available after analogue-to-digital conversion (ADC), cf. Fig. 1; $f_s = 1/T_s$ denotes the sampling frequency. The available data $\hat{y}[n]$ is assumed to be related to $y[n]$ according to $\hat{y}[n] = y[n] + e[n]$, where the ADC errors $e[n]$ are modeled by a stationary, zero mean random process with known autocovariance function $R(n-m) = \mathrm{E}(e[n]e[m])$. The relation between $y_c(t)$ and $x_c(t)$ shall be described by an LTI system according to

$$y_c(t) = (h_{\boldsymbol{\theta}} * x_c)(t) \quad , \tag{1}$$

where $h_{\boldsymbol{\theta}}(t)$ denotes its impulse response. $h_{\boldsymbol{\theta}}(t)$ is parameterized by the parameter vector $\boldsymbol{\theta} = (a_0, a_1, \ldots, b_0, b_1, \ldots)^{\mathrm{T}}$ which determines its system function according to

$$H_{\boldsymbol{\theta}}(s) = \sum b_i s^i \Big/ \sum a_j s^j \ . \tag{2}$$

We assume that an estimate $\hat{\boldsymbol{\theta}}$ including its variance-covariance matrix $\boldsymbol{U}_{\hat{\boldsymbol{\theta}}}$ is available. This requires that system identification [8] using an adequate dynamic model has already been carried out, and that uncertainties in line with the GUM have been assigned to the obtained parameter estimates, cf. [9].

We consider the application of a digital filter $g_{\boldsymbol{\mu}(\boldsymbol{\theta})}[n]$ with system function

$$G_{\boldsymbol{\mu}}(z) = \sum d_i z^{-i} \Big/ \sum c_j z^{-j} \tag{3}$$

to the sensor output signal $y[n]$, where $\boldsymbol{\mu}(\boldsymbol{\theta})$ indicates a mapping from the parameters $\boldsymbol{\theta}$ of $h_{\boldsymbol{\theta}}(t)$ to the parameters $\boldsymbol{\mu} = (c_0, c_1, \ldots, d_0, d_1, \ldots)$ of the digital filter (3). The mapping $\boldsymbol{\mu}(\boldsymbol{\theta})$ is characterized by the way the digital filter is constructed. For instance, $g_{\boldsymbol{\mu}(\boldsymbol{\theta})}[n]$ could result from an adjustment of its frequency response to the reciprocal frequency response of $h_{\boldsymbol{\theta}}(t)$ within some chosen frequency range, followed by a low-pass filter. Note that the digital filter (3) may contain a different number of parameters than the continuous-time system function (2). The possibly time-shifted discrete-time estimates $\hat{x}[n]$ of $x[n] = x_c(n \cdot T_s)$ are then obtained by the application of the digital filter $g_{\boldsymbol{\mu}(\boldsymbol{\theta})}[n]$ to the sensor output data

$$\hat{x}[n - n_0] = \left(g_{\boldsymbol{\mu}(\hat{\boldsymbol{\theta}})} * \hat{y}\right)[n] \, , \tag{4}$$

where n_0 denotes the known time shift. The task to be considered is that of determining the uncertainties $u(\hat{x}[n - n_0])$ associated with these estimates in line with GUM or GUM S1, respectively. When the digital filter does not perfectly compensate the dynamic behavior of the sensor, remaining dynamic errors need to be taken into account. To this end, some knowledge about the measurand is required, and we assume knowledge about an upper bound $\overline{X}(\omega)$ on the modulus of the continuous-time Fourier transform

$$\left| X_c(\omega) \right| \leq \overline{X}(\omega) \tag{5}$$

of the measurand. Note that when a static analysis is applied, the 'filter' $g_{\boldsymbol{\mu}(\boldsymbol{\theta})}[n]$ would simply scale the output signal according to the static properties of the sensor.

3. UNCERTAINTY EVALUATION

The model in terms of the GUM relating the measurand $x[n] = x_c(n \cdot T_s)$ to its input quantities is given by

$$x[n - n_0] = \left(g_{\boldsymbol{\mu}(\boldsymbol{\theta})} * y\right)[n] + \Delta[n] \, , \tag{6}$$

where $\Delta[n]$ denotes the dynamic error, and n_0 a possible (and known) time shift implied by the construction of the compensation filter $g_{\boldsymbol{\mu}(\boldsymbol{\theta})}[n]$, cf. [6]. The parameters $\boldsymbol{\theta}$, the sensor output signal $y[n] = y_c(n \cdot T_s)$ as well as the dynamic error $\Delta[n]$ are considered as the input quantities.

According to GUM, estimates $\hat{x}[n - n_0]$ of $x[n - n_0]$ are obtained by inserting the estimates for $\boldsymbol{\theta}$ (and hence for $\boldsymbol{\mu}(\boldsymbol{\theta})$), for $y[n]$ and for $\Delta[n]$ into (6). The estimates for $\boldsymbol{\theta}$ and $y[n]$ are given by $\hat{\boldsymbol{\theta}}$ and the data $\hat{y}[n]$. For the former, $\boldsymbol{U}_{\hat{\boldsymbol{\theta}}}$ contains the required uncertainties, while for the latter the uncertainties $u(\hat{y}[n], \hat{y}[m]) = R(n - m)$ are assigned in accordance with the autocovariance function $R(n - m) = \mathrm{E}(e[n]e[m])$ of the noise process modeling ADC errors, cf. [6]. Regarding the (unknown) dynamic error, the approximate bound

$$\left| \Delta[n] \right| \lesssim \frac{1}{2\pi} \int_{-\pi f_s}^{\pi f_s} \left| e^{j\omega n_0 / f_s} G_{\boldsymbol{\mu}(\hat{\boldsymbol{\theta}})}(e^{j\omega / f_s}) H_{\hat{\boldsymbol{\theta}}}(j\omega) - 1 \right| \cdot \overline{X}(\omega) \, \mathrm{d}\omega \tag{7}$$

$$=: \gamma \quad .$$

is employed which can be derived by application of Fourier techniques. This bound, obtained for the discrete-time processing applied here, is similar to a bound given in [10] for continuous-time processing. We assign a uniform PDF for $\Delta[n]$ on $[-\gamma,\gamma]$. As a consequence, the estimate $\hat{\Delta}[n]=0$ of the dynamic error is obtained together with $u(\hat{\Delta}[n])=\gamma/\sqrt{3}$ uniformly for all n.

The uncertainties associated with the $\hat{x}[n-n_0]$ are then according to (6) obtained as

$$\begin{aligned} u^2(\hat{x}[n-n_0]) &= u^2\left\{\left(g_{\mu(\hat{\theta})} * \hat{y}\right)[n]\right\} + u^2(\hat{\Delta}[n]) \\ &= u^2\left\{\left(g_{\mu(\hat{\theta})} * \hat{y}\right)[n]\right\} + \gamma^2/3 \end{aligned} , \qquad (8)$$

where γ is given by (7).

The first term on the right-hand side of (8) accounts for the uncertainty of the output signal and the estimates of the system parameters. It can be evaluated by the application of the Monte Carlo procedure described in GUM S1, after assigning (multivariate) Gaussian PDFs to the input quantities $\boldsymbol{\theta}$ and $y[n]$. Note that when applying GUM S1, an estimate slightly different from that in (4) may be obtained. Application of GUM S1 is possible only off-line. Nevertheless, from the perspective of a user it is a convenient way.

When the digital filter $g_{\mu(\theta)}$ is of FIR-type, results identical to a GUM S1 application can be obtained by a real-time calculation scheme [6]. For an IIR-type filter, a real-time calculation scheme has been made possible utilizing a state-space representation [11]. However, the latter calculation scheme is based on a first-order Taylor series expansion which may not yield appropriate results in all cases since an IIR filter depends nonlinearly on its parameters.

The second term on the right-hand side of (8) accounts for the dynamic errors, i.e. $\gamma^2/3$ is the (squared) uncertainty component which represents our lack of knowledge about the actual value of the dynamic error. Note that the dynamic error is a systematic error; when repeating the measurement and the analysis for the same measurand, the same (unknown) dynamic error will emerge.

It is worth briefly discussing the bound (7). For large frequencies, the term $\left|e^{j\omega n_0/f_s} G_{\mu(\hat{\theta})}(e^{j\omega/f_s}) H_{\hat{\theta}}(j\omega) - 1\right|$ is expected to approach 1, and the integral is kept small by the decay of the upper bound $\overline{X}(\omega)$ of the spectrum of the measurand. In order to keep the integral small for small frequencies on the other hand, the term $\left|e^{j\omega n_0/f_s} G_{\mu(\hat{\theta})}(e^{j\omega/f_s}) H_{\hat{\theta}}(j\omega) - 1\right|$ itself needs to be small which requires that for these frequencies the digital filter compensates the dynamic behavior of the sensor. Note that when a static analysis is applied and $\overline{X}(\omega)$ has considerable content for non-zero frequencies, a large bound and hence a large uncertainty will result. Note further that the uncertainty associated with the dynamic error can be controlled by the design of the digital filter, and this can be used to design an uncertainty-optimal filter.

4. EXAMPLE

We illustrate the uncertainty evaluation for the second-order model

$$H(j\omega) = \frac{s_0}{1+(j\omega/\omega_0)^2 + 2j\delta\omega/\omega_0}, \qquad (9)$$

for which we assume the parameter estimates $\hat{f}_0 = 50 \pm 0.5$ kHz, $\hat{\delta} = (2 \pm 0.2)\cdot 0.01$ and $\hat{s}_0 = 1 \pm 0.001$. The sampling frequency was chosen as 10 times the resonance frequency $f_0 = \omega_0/2\pi$ and the measurand as a Gaussian-like signal $x_c(t)$. Fig. 2 shows the magnitude response $|H(j\omega)|$ of this model together with the (normalized) spectrum of the measurand. In addition, the magnitude response $|G(e^{j\omega/f_s})|$ of the digital filter applied to estimate the measurand from the output of the LTI system is also shown.

The measurand $x_c(t)$ was passed through the LTI system (9), and the output signal was subsequently discretized and corrupted by white noise (with a standard deviation of 0.1% relative to the signal's maximum). Using this output signal, the measurand was then estimated by the application of the two compensation filters shown in Fig. 2. Both filters were constructed as the cascade of an approximate inverse filter of FIR type derived from the parameter estimates of the LTI system and (different) low-pass filters, cf. [6] for details on constructing the inverse filters. The uncertainties associated with the resulting estimates were then determined according to (8). As upper bound for the spectrum of the measurand we employed the magnitude of its actual spectrum. Fig. 3 shows the resulting uncertainties.

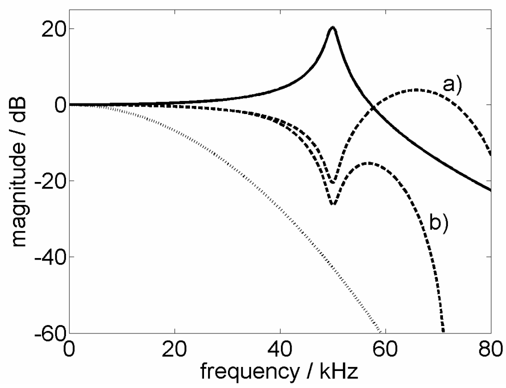

Fig. 2. Magnitude response of the LTI system (solid line) and of the two compensation filters (dashed lines), together with the (normalized) magnitude of the measurand (dotted).

The difference between the two compensation filters is that the first one (filter a) in Fig. 2) compensates the LTI system also for larger frequencies. This leads here to smaller uncertainties, but this holds not in general and it depends –

among other things – on the size of the noise. For large noise, it may be desirable to strongly suppress the noise also for smaller frequencies, thereby enlarging the dynamic error. The key point is that the combined uncertainty (8) accounts for both effects and it therefore allows for the construction of a digital filter which yields an estimate with minimum uncertainty.

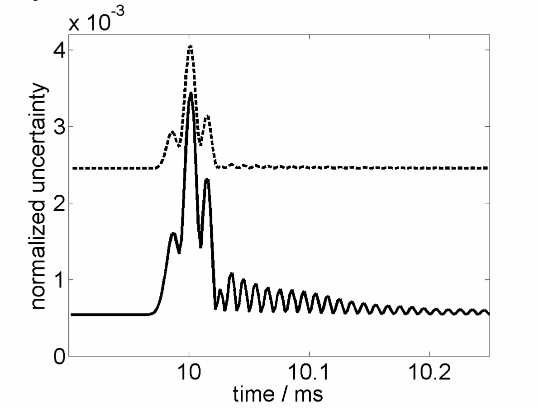

Fig. 3. Normalized uncertainty obtained after applying the compensation filters from Fig. 2. Lower curve: filter a), upper curve: filter b). Uncertainties were normalized by the maximum amplitude of the input signal.

5. CONCLUSIONS

Evaluation of uncertainties in line with the GUM has been considered for the task of estimating the input signal of an LTI system given its discretized output signal. It was assumed that the input signal is estimated by the application of a digital filter, and that an upper bound on the magnitude of the spectrum of the measurand is available. It has been shown that for this scenario the framework of the GUM can be applied for uncertainty evaluation, also in the presence of dynamic errors. A corresponding calculation scheme has been proposed and illustrated by an example. The uncertainty calculation scheme may then be used for the construction of a digital compensation filter which results in an estimate with minimum uncertainty.

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