XIX IMEKO World Congress
Fundamental and Applied Metrology
September 6−11, 2009, Lisbon, Portugal

COMPARATIVE ANALYSIS OF THE MEASUREMENT UNCERTAINTY OF THE DEFORMATION COEFFICIENT OF A PRESSURE BALANCE USING THE GUM APPROACH AND MONTE CARLO SIMULATION METHODS

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Abstract − Uncertainty estimation results are presented for the intercept and slope of the deformation coefficient of a piston-cylinder set of a Budenberg pressure balance operating in the range of 16100 psi (111 MPa) by using the GUM approach and Monte Carlo simulation methods. A study of the influence of the correlation between quantities that define the deformation coefficient on the value of its uncertainty is shown. Monte Carlo simulations are done by using the commercial software *Crystal Ball®*.

Keywords: Calibration, Uncertainty, Monte Carlo.

1. INTRODUCTION

The Guide to the Expression of Uncertainty in Measurement (GUM) [1] is an orientation document published by ISO (*International Organization for Standardization*) that attempts to harmonize the procedures for measurement uncertainty estimation. Since its publication, this document has been continually used as a general guide by several metrology institutes around the world. However, the measurement uncertainty estimation approach by using the law of propagation of uncertainties used in the GUM has some limitations, being valid only in some situations. This approach is valid only if the following assumptions are considered: (a) The model used for the measurand calculation should have linear behavior. When the model presents non linear behavior, the approximation made by the GUM approach that truncates the Taylor series expanded model in the first term is not sufficient anymore to estimate the measurand uncertainty. (b) All the probability distribution functions (PDFs) of the source quantities should be considered symmetrical, which is not applicable in some cases of the electrical, optics and acoustics metrology. (c) The Central Limit Theorem validity is admitted, that states that the PDF resulted from the convolution of a large number of PDFs has a normal behavior, that is, the PDF for the measurand is assumed to have a normal behavior, which is not the case in all situations. (d) After obtaining the standard uncertainty by the law of propagation of uncertainties, the GUM approach uses the Welch-Satterthwaite formula to calculate the number of effective degrees of freedom, which is used to estimate an expanded uncertainty. This procedure approximates the measurand PDF of a t-Student distribution, which is not the case in all situations.

In order to overcome the limitations of the GUM approach, propagation of distributions methods have been applied to metrology. The propagation of distributions is a generalization of the GUM approach, involving richer information than that of simple propagation of uncertainties. Propagation of distributions involves the convolution of the source PDFs, which can be done in three different ways: a) analytical integration, b) numerical integration or c) numerical simulation. The GUM supplement 1 [2] presents basic procedures on how to use Monte Carlo numerical simulation methods to propagate distributions in metrology.

In this work, uncertainty estimation results are presented for the deformation coefficient of a piston-cylinder set of a Budenberg pressure balance operating in the range of 16100 psi (111 MPa). Uncertainties are estimated by applying the GUM approach and Monte Carlo simulation methods. A study of the influence of the correlation between quantities that define the deformation coefficient on the value of its uncertainty is shown. Monte Carlo simulations are done by using the commercial software *Crystal Ball®* [3].

2. EXPERIMENTAL

The pressure balance is a pressure measurement equipment based on the equilibrium between two forces [4]. The first refers to a set of masses that act on the top of the balance piston under the influence of gravity. The second is the one that acts on the base of the piston-cylinder set and is defined by multiplying the pressure exerted by the fluid and the cross sectional area of the piston-cylinder set.

The equation that defines the pressure measurement in a pressure balance is (1):

ISBN 978-963-88410-0-1

$$p=\frac{\left[m_p\cdot\left(1-\frac{\rho_a}{\rho_{mp}}\right)\sum m\cdot\left(1-\frac{\rho_a}{\rho_m}\right)\right]\cdot g_l+\sigma C}{A_{0.20}[1+(\alpha_c+\alpha_p)\cdot(t-20)]\cdot(1+\lambda p_n)}+\rho_{oil}g_l\Delta h \tag{1}$$

Where:

m_p = mass of the piston (kg);
ρ_a = specific mass of the air (kg/m³);
ρ_{mp} = specific mass of the material of the piston (kg/m³);
Σm = sum of the masses on the top of the piston (kg);
ρ_m = specific mass of the material of the masses (kg/m³);
g_l = local gravity acceleration (m/s²);
σ = surface tension of the test fluid (N/m);
C = length of the circumference of the piston (m);
$A_{0.20}$ = cross sectional area of the piston-cylinder (m²);
α_c,α_p = linear expansion coefficient of the piston-cylinder set (°C^{-1});
t = temperature (°C);
λ = deformation coefficient of the piston-cylinder set (Pa^{-1});
p_n = nominal measurement pressure (Pa);
ρ_{oil} = specific mass of the oil (kg/m³);
Δh = difference in height between the piston base and the point where the pressure is measured.

Calibration of the pressure balance was done using the cross-floating method, which is based on the equilibrium between the pressures measured by the test and the reference. In this way, the calibration can be described by equation (2):

$$p_{REF}=\frac{\sum m\cdot\left(1-\frac{\rho_a}{\rho_m}\right)\cdot g_l+\sigma C}{A_{0.20}[1+(\alpha_c+\alpha_p)\cdot(t-20)]\cdot(1+\lambda p_n)}+\rho_{oil}g_l\Delta h \tag{2}$$

Where:
p_{REF} = reference pressure (Pa);
Σm = sum of the masses on the top of the tested piston (kg);
ρ_a = specific mass of the air (kg/m³);
ρ_m = specific mass of the material of the masses (kg/m³);
g_l = local gravity acceleration (m/s²);
σ = surface tension of the test fluid (N/m);
C = length of the circumference of the tested piston (m);
$A_{0.20}$ = cross sectional area of the tested piston-cylinder set (m²);
α_c,α_p = linear expansion coefficient of the tested piston-cylinder set (°C^{-1});
t = temperature (°C);
λ = deformation coefficient of the tested piston-cylinder set (Pa^{-1});
p_n = nominal measurement pressure (Pa);
ρ_{oil} = specific mass of the oil (kg/m³);
Δh = the height difference between the reference and test piston floating level (m). This can be positive or negative if the reference is above or under the test, respectively.

All variables are known in this equation (2), except for the effective area (A_e) of the test, which is calculated by equation (3) for each point of nominal equilibrium pressure defined as being 10% of the scale range of the instrument to be calibrated.

$$A_e=A_{0.20}\cdot(1+\lambda p_n) \tag{3}$$

Operating (3) gives (3a):

$$A_e=A_{0.20}+A_{0.20}\lambda p_n=a+bp_n \tag{3a}$$

Where a and b are respectively the intercept and the slope of a linear regression line. In this way, the deformation coefficient (λ) of the piston-cylinder set can be determined by equation (4):

$$\lambda=\frac{b}{a} \tag{4}$$

In this work, the pressure balance was calibrated in eleven nominal pressure points. According to the GUM [1], the uncertainties due to the intercept and slope, as well as their correlation coefficient are estimated by equations (5), (6) and (7), respectively.

$$u_a=\sqrt{\frac{s^2\sum p_n^2}{D}} \tag{5}$$

$$u_b=\sqrt{n_t\frac{s^2}{D}} \tag{6}$$

$$r_{(a,b)}=-\frac{\sum p_n}{\sqrt{n_t\sum p_n{}^2}} \tag{7}$$

Where n_t is the total number of points of aquired, s^2 is the variance obtained in equation (8) and D the matrix determinant obtained in equation (9):

$$s^2=\frac{\sum\Delta^2}{n_t-2} \tag{8}$$

$$D=n_t\sum p_n{}^2-(\sum p_n)^2 \tag{9}$$

The value of Δ in (8) is the line deviation that defines the variation of the effective cross sectional area of the piston-cylinder set of a pressure balance as a function of the nominal pressure of calibration. In this way, according to the GUM, the uncertainty components due to the slope and intercept define the uncertainty of the deformation coefficient of the piston-cylinder set as (10):

$$u_\lambda(a,b)=\sqrt{c_{ia}^2\cdot u_a^2+c_{ib}^2\cdot u_b^2+2\cdot c_{ia}\cdot c_{ib}\cdot u_a\cdot u_b\cdot r_{a,b}} \tag{10}$$

Where the sensitivity coefficients are:

$$c_{ia}=\frac{\partial\lambda}{\partial a}=-\frac{b}{a^2} \tag{11}$$

and

$$c_{ib} = \frac{\partial \lambda}{\partial b} = \frac{1}{a} \quad (12)$$

3. RESULTS AND DISCUSSION

Table 1 shows the results of effective area as a function of the nominal pressure.

Table 1 – Effective area results as a function of the nominal pressure.

p_n (psi)	A_e (m²)
1100	4.031449E-06
1800	4.031840E-06
3400	4.032034E-06
5000	4.032169E-06
6600	4.032346E-06
8200	4.032484E-06
9800	4.032661E-06
11400	4.032844E-06
13000	4.033068E-06
14600	4.033101E-06
16100	4.033248E-06

Plotting the data from Table 1 values for the slope and intercept of the adjusted line could be obtained. Values for s^2_a, u_a, s^2_b, u_b, $r_{a,b}$, the line regression coefficient and the deformation coefficient of the piston-cylinder set were also calculated and are shown in Table 2.

Table 2 – Values for the slope, intercept, regression coefficient, s^2_a, u_a, s^2_b, u_b, $r_{a,b}$, and the deformation coefficient of the piston-cylinder set.

Parameter	Value	Unit
Intercept (a)	4.03157E-06	m²
Slope (b)	1.09189E-13	m²/psi
Regression coefficient (r^2)	0.98	------
s^2_a	3.71264E-21	m^4
s^2_b	4.00705E-29	m^4/psi^2
u_a	6.09314E-11	m²
u_b	6.33012E-15	m²/psi
Correlation coefficient ($r_{a,b}$)	-0.86	-----
Deformation coefficient (λ)	2.7E-08	psi^{-1}
$u_c\lambda$	1.57E-09	psi^{-1}

Fig.1 shows the results of the uncertainty budget of the deformation coefficient of the piston-cylinder set by using the GUM approach, as well as the its combined uncertainty $u_c\lambda$.

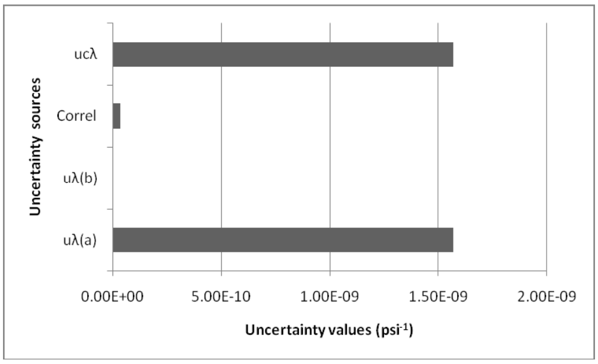

Fig. 1 – Uncertainty budget of the deformation coefficient of the piston-cylinder set by using the GUM approach.

Table 3 shows the uncertainty budget of the deformation coefficient assuming different values for $r_{a,b}$, from -0.86 (experimental value) to -1.0 (simulated value).

Table 3 – Uncertainty budget results of the deformation coefficient of the piston-cylinder set by using the GUM approach.

Source	$r_{a,b}$	
	-0.86	-1.0
$u\lambda$ (a)	99.96%	99.95%
$u\lambda$ (b)	0.00%	0.00%
$r_{a,b}$	0.04%	0.05%
$uc\lambda$	100.00%	100.00%

Table 4 shows the values of the superior and inferior limits of the expanded uncertainties (95% confidence interval) of the deformation coefficient of the piston-cylinder estimated by the GUM approach and by the Monte Carlo simulation using 100000 runs. Different values for the intensity of the correlation between the parameters that define the relationship between the effective area of the piston-cylinder set and the nominal calibration pressure area assumed.

Table 4 – Superior (d_{sup}) and inferior (d_{inf}) limits of 95% confidence intervals of the expanded uncertainties estimated by the GUM approach and the Monte Carlo simulation (MCS).

$r_{a,b}$	Limits	GUM	MCS
-0.86	d_{inf}	2.353E-08	2.394E-08
	d_{sup}	3.064E-08	3.010E-08
-1.00	d_{inf}	2.407E-08	2.407E-08
	d_{sup}	3.064E-08	3.021E-08
0.86	d_{inf}	2.402E-08	2.402E-08
	d_{sup}	3.063E-08	3.019E-08
1.00	d_{inf}	2.404E-08	2.404E-08
	d_{sup}	3.063E-08	3.016E-08

According to the criteria of equivalence between coverage intervals used in the GUM supplement 1 [2] (GUM validation by the Monte Carlo method) one can admit a value for the expanded uncertainty with two significant numbers, writing it in the form $c \times 10^l$, where c and l are integers. In the case of this work, the expanded uncertainty obtained from the standard uncertainty of Table 2, can be written as 31×10^{-10} psi^{-1}. In this way, the numerical tolerance (defined as $\delta = 0.5 \times 10^l$) in this case is $\delta = 0.5 \times 10^{-10}$. According to the GUM supplement 1, the coverage intervals of different distributions are considered to be equivalent if the differences between their superior and inferior limits are lower than the numerical tolerance associated with the standard uncertainty (δ).

Figs. 2 and 3 show that all differences between the limits of the intervals obtained by the two methods are higher than the numerical tolerance (δ), indicated as a line.

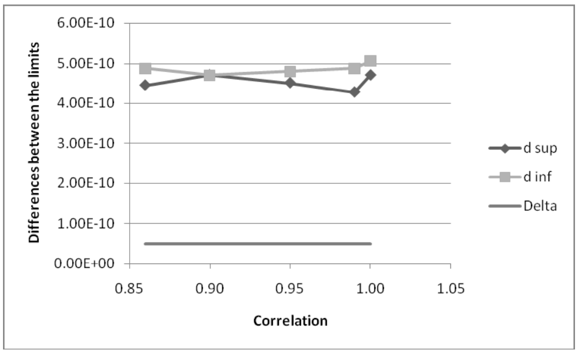

Fig. 2 – Differences between the limits of the coverage intervals with the numerical tolerance (δ) considering a positive correlation coefficient.

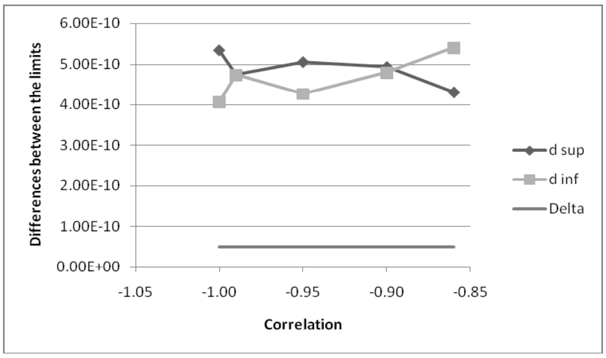

Fig. 3 – Differences between the limits of the coverage intervals with the numerical tolerance (δ) considering a negative correlation coefficient.

4. CONCLUSIONS

The uncertainty component that was more relevant to the overall uncertainty estimation of the deformation coefficient of the piston-cylinder set was due to the intercept parameter, associated with the relationship between the effective area and the nominal calibration pressure, even for higher correlations, the impact to the uncertainty value is minimal.

According to the criteria of equivalence between coverage intervals used in the GUM supplement 1 [2] (GUM validation by the Monte Carlo method) one can conclude that the coverage intervals of the distributions obtained for both estimation methods (GUM approach and Monte Carlo simulation) were not equivalents.

The Monte Carlo simulation method showed to be the more appropriate method for the uncertainty estimation of the deformation coefficient of the piston-cylinder set of a pressure balance.

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