

SHIFTED UP COSINE FUNCTION AS MODEL OF PROBABILITY DISTRIBUTION

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Abstract – The shifted up one period of cosine function with field under it normalized to 1 is proposed to be used as the unconventional model of probability density function (PDF). It could also approximate Normal probability distribution in the range $\pm 2,5$ standard deviation with accuracy of about $\pm 0,02$, which is fully acceptable in the evaluation of measurement uncertainty. In this paper the properties of the above cosine based PDF are considered. The possibility of its applications in the routine data assessment and in virtual instruments with automatic uncertainty calculations is recommended.

Keywords: probability density function PDF, cosines approximation of Normal distribution

1. INTRODUCTION

Normal (Gaussian) probability density function – PDF is the most commonly applied as a model of error and uncertainty distributions including international GUM recommendations [1]. This application is valid due to the assumption that a large number of independent factors are randomly influencing upon the measured object and/or the measuring chain. In practice, one of the inconveniences of applying Normal PDF is non limited range of the measurand values \pm infinity. In measuring experiments there is a limited number of influencing variables and there are physical limits of collected data values. The additional information that data values are limited allows for using a function of limited width as a model of distribution, which estimates the data dispersion better than unlimited Gauss PDF. This difficulty is not solved good enough by the Gauss distribution of cut off both end tails and of normalized left area under the curve to 1. The description of such tails' truncated Gauss function is rather complex. Then, non-Gaussian different distributions are also applied to uncertainty evaluations. Some of them, which are recommended for uncertainty evaluation by Monte Carlo method, are given in Table 1 of Supplement 1 to GUM [2]. More accurate estimators than the mean value may be also obtained for non-Gaussian distributions [4, 9].

No matter if the Monte Carlo method or the convolution based method is used, the simplification of Normal and t-Student distributions are a great assistance to reduce a number of mathematical operations as in both cases data tables of consume a lot of memory space. Furthermore the authors started developing instruments, where the automatic function of uncertainty calculation is implemented [5, 6]. Different distributions have to be applied for online processing of signal uncertainty. In smaller and simpler intelligent transducers and even in instruments applying DSP the memory is rather limited. Generation of simple PDF model, which approximates Normal distribution, would be considered very helpful in this case.

One of us (ZLW) is suggesting to use as a distribution model of random data of limited values other than Gauss smooth function, e.g. the shifted up cosine in the range of $-\pi$ to $+\pi$. In the simpler case this function may touch horizontal axis "x" – see Fig. 1 or even may be located above this axis. When theoretical bases of that unconventional model were elaborated by us, a mention of similar type model was found in the short paper by H. Green and D. Rabb published long ago in 1961 [7]. Green proposed arbitrarily to use a particular non-optimal shifted up cosine function in the range $\pm\pi$ as an approximation of Normal distribution. There are neither method how to find the cosine based function of optimal parameters nor the accuracy of approximation has been elaborated. That may be why Green's proposal was not later developed in statistical and metrology publications. Today's computer based possibilities of calculations do not limit estimating the best parameters of any type of PDF curve proposed for approximation of the measurement data distribution.

2. THEORETICAL BASES

2.1. Formula of cosine-based PDF

Proposed is unconventional model of probability density distribution function in the generalized form

$$f(x) = B + A \cos 2\pi \frac{x}{X_T}, \quad (1)$$

where: x – value of observations, $f(x) > 0$, X_T – range equal to one period of cosines ($1/X_T$ as frequency in time functions), A, B – constant parameters.

Because of the constant value B the function $f(x)$ of (1) was named as **shifted up cosine** function with proposed symbol $+COS$. As model of the PDF distribution is valid for $f(x) > 0$ only and its field is equal 1, then the function $+COS$ could be applied in the range of one period $\pm(X \leq 0,5X_T)$.

Function $f(x)$ from (1) generally has three independent parameters A, B, X . So, to find them three relations are needed, i.e.:

- field under curve in its range $S=1$, or $S < 1$, e.g. as for Gauss curve in the same range,
- range of approximation $X_{APPROX} < X$ or point to cross Gauss function or particular parameter of $+COS$ function, e.g.: $\pm\sigma, \pm2\sigma, \dots$ similarly as for Gauss
- minimum difference between two curves according to some criterion: LSM, minimum sum of difference modules, Chebyshev criterion and others.

2.2. Cosine shifted up by $B > A$

In the case when one period of cosine function is shifted by $B > A$, the field under curve $S=2XB$. From $S=1$ half range X of cosine function is

$$X=1/(2B). \quad (2)$$

The function (1) can be expressed now as

$$f(x) = B + A \cos(2\pi xB). \quad (3)$$

Function (3) has two independent parameters, i.e. A and B (or A and X).

When $X=1/2B$ the cumulative distribution CPDF as integral of (3) is described by following relation

$$F(x) = \int f(x) dx = Bx + \frac{1}{2\pi} \frac{A}{B} \sin \pi \frac{x}{X} + const \quad (4)$$

for $-X \leq x \leq X$.

Functions (3), (4) could be taken for approximation of Normal distributions, but only if higher accuracy than in particular case $A=B$ is needed. For $A=B$ value of $X=1/(2A)$ is higher than for $A > B$ and then (4) it is

$$F(x) = \frac{1}{2} \left(\frac{x}{X} + 1 \right) + \frac{1}{2\pi} \sin \pi \frac{x}{X} \Big|_{-X \leq x \leq X} \quad (4a)$$

Some values of $F(x)$ for $A=B$ and then $X=1/2A$ are given below in Table 1.

Table 1. Values of COS^2 model CPDF.

x/X	-1	-0.9	-0.8	-0.7	-0.6	-0.5
$F(x)$	0	0.001	0.006	0.021	0.049	0.091
x/X	-0.4	-0.3	-0.2	-0.1	0	
$F(x)$	0.149	0.221	0.307	0.401	0.5	
x/X	0.4	0.3	0.2	0.1	0	
$F(x)$	0.851	0.780	0.694	0.599	0.5	
x/X	1	0.9	0.8	0.7	0.6	0.5
$F(x)$	1	0.999	0.994	0.979	0.951	0.909

2.3. Cosine shifted up by its amplitude A

For model of PDF is proposed the simplest shifted up cosine function when $B=A$. In two its minimum it is tangential to x axe for $x=\pm 0.5X_T$ and from (1) is

$$f(x) = A \left(1 + \cos \pi \frac{x}{X} \right). \quad (5)$$

To obtain cumulative distribution CPDF the range of $f(x)$ integration is $\pm X$. Field for this range $S=1$ and after integration of (5): $A \cdot 2X=1$ and $2\sin(x=\pm X) = 0$.

Then:

$$X=1/(2A), \quad (6)$$

and

$$f(x) = A(1 + \cos 2\pi Ax) \equiv 2A \cos^2 \pi Ax. \quad (7)$$

Parameters A and X are mutually dependent as in (6) and it is enough to have one of them as given or from histogram of data or from approximation of Normal PDF by the above function acc. to a given rule.

Function $+COS$ of (7) with $A=B$ is named as COS^2 may be so chosen to passing the given point. If such $+COS$ function is e.g. covering the top point of Gauss distribution of standard deviation σ – see curve 2 on fig 1, then the amplitude of its cosine component is:

$$A = \frac{1}{2} \frac{1}{\sigma \sqrt{2\pi}} \quad (7a)$$

and as the field under curve (2) is $S=1$, then from (3) half of cosine range

$$X = \sigma \sqrt{2\pi} \approx 2.507 \sigma.$$

For normalized Gauss distribution, i.e. when its standard deviation $\sigma=1$ it is

$$A = \frac{1}{2} \frac{1}{\sqrt{2\pi}} = 0,1995 \quad \text{and} \quad X = \sqrt{2\pi} \approx 2,507.$$

2.4. Standard deviation of $B+Acos(\cdot)$ distribution

Standard deviation σ_{+cos} of the PDF model $+COS$ function in general case (6) is

$$\sigma_{+cos} = \sqrt{\int_{-X}^{+X} x^2 (B + A \cos 2\pi xB) dx}. \quad (8)$$

In the solution of (7) the generalized formula of $\int x^n \cos cx dx$ solution adopted for x^2 , is applied:

$$\int x^2 \cos cx dx = \frac{\sin cx}{c} \left(x^2 - \frac{2}{c^2} \right) + \frac{2x \cos cx}{c^2}, \quad (8a)$$

where: $c=2\pi B$ and for $x=X$, $B=1/2X$ or $X=1/2B$, $B \geq A$.

Two parameters are independent and solution of (8) could be presented in a double form as

$$\sigma_{+cos} = X \sqrt{\frac{1}{3} - \frac{4}{\pi^2}} AX = \frac{1}{2B} \sqrt{\frac{1}{3} - \frac{2}{\pi^2}} \frac{A}{B}. \quad (9)$$

Solution is possible only for limited ratio of A/B as

$$\frac{A}{B} \leq \frac{\pi^2}{6} \approx 1.78.$$

In the particular case of COS^2 PDF shifted up cosine is tangential to axe x . From (3) $A=B=1/2X$ and then

$$\sigma_{+\cos} = X \sqrt{\frac{1}{3} - \frac{2}{\pi^2}} = \frac{1}{2A} 0.3615. \quad (9a)$$

If that standard deviation $\sigma_{+\cos}$ is given, it is possible to find mutually joint values A and X . For example if $\sigma_{+\cos} = 1$ is: $A=0,1808$, $X= 2.766$. Or in opposite, for a given range X , e.g. $X = 2,5$ from (9a) $\sigma_{+\cos} = 0,9060$.

For extended uncertainty calculations of $+COS$ PDF cover factors k_{PC} for given P and large $n>10$ are put below in Table 2.

Table 2. Extended uncertainty cover factors of $+COS$ PDF

P [%]	50	68.3	90	95	99	99.7	100
$k_{PC}(x/X)$	0.265	0.385	0.596	0.683	0.816	0.878	1

For smaller $n \leq 10$ additional extension than k_{PC} in Table 2, nearer to the Student-Gosset PDF values, should be applied.

3. APPROXIMATION of NORMAL DISTRIBUTION by FUNCTION + COS

In Fig 1a are shown: curve 1 – normalized Gauss PDF $N(0,1)$, i.e. $f_G(x) = \frac{1}{\sqrt{2\pi}} e^{-0.5x^2}$ of $\bar{x} = 0$, $\sigma=1$ and three $+COS$ functions $f(x)=A(1+\cos 2\pi Ax)$: curve 2 – of $A = (2\sigma\sqrt{2\pi})^{-1}$ going through the top point of 1, curve 3 – of standard deviation $\sigma_{+\cos}=1$ and curve 4 – $f_{GR}(x) = \frac{1}{2\pi}(1 + \cos x)$ of $-\pi < x < +\pi$ proposed by Green [7]. Fields S under curves are 1.

Differences Δ_{PDF} between each of above COS^2 distributions and Gauss of $\sigma = 1$ are on Fig 2a and of their cumulative distributions CPDF Δ_{CPDF} on Fig 2b.

Values of main parameters of above three functions are in columns 2 to 4 of Table 3. From Fig 2a,b and data of Table 3 it is possible to conclude that difference Δ_{PDF} of COS^2 distribution 2 and normalized Gauss PDF as well Δ_{CPDF} of curves 2 and 3 do not exceed range $\pm 2.1\%$. Δ_{PDF} of Gauss and $+COS$ function 3, both of standard deviation $\sigma=1$, are changed in the little broader range $(+2,8\%, -3,7\%)$.

Even this accuracy could be fully accepted to the most uncertainty type A evaluations. In column 4 are also given for comparison parameters of function $f_{GR}(x)$ proposed by Green in [7]. It has $A=1/(2\pi)$, standard deviation $\sigma_{+\cos} \approx 1,14$.

Differences Δ_{PDF} and Δ_{CPDF} of Green function $f_{GR}(x)$ and normalized Gauss distributions are much higher (up to $+4.5\%$, -8.1%) then of top point function 3 and is not acceptable for the most research and technical measurements.

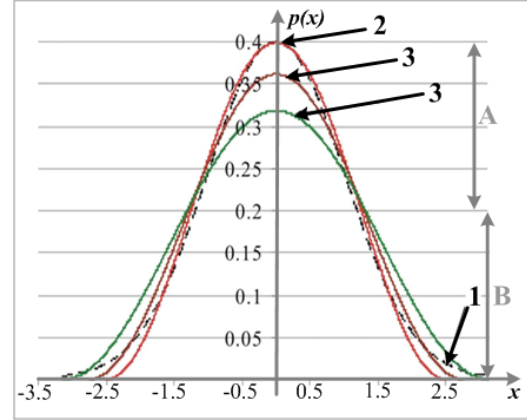


Fig. 1. PDF distributions: 1 – Normalized Gauss $N(0,1)$ (i.e. $\sigma=1$) and three tangential to axe x functions $A(1+\cos 2\pi Ax)$: 2 – passing through the top point of Gauss PDF, 3 – of standard deviation $\sigma_{+\cos}=1$, 4 – Green proposal $f_{GR}(x)$ [7].

For approximation of the Gauss PDF two other $+COS$ functions of the single optimal parameter $A=B$ are given in columns 5 and 6 of Table 3. Value A of curve 5 is calculated from differences Δ_{PDF} by commonly known least square method LSM and of curve 6 by LMM method – min sum of modules.

Differences Δ_{PDF} and Δ_{CPDF} of distributions 5 and 6 together with 2 in relation to Gauss PDF compared are in Fig 3a,b. The accuracy of Gauss approximation by functions 5, 6 in the full period of cosine are very near to the accuracy of the top point curve 2.

For shorter range of approximation it is possible to obtain a little better accuracy. Dependence of optimal values of A and of half range X of the $+COS$ on the range of approximation X_{APPROX} is given on Fig 4a. Changes of Δ_{PDF} and Δ_{CPDF} main parameters are on Fig. 4b,c.

Mean, extreme values and standard deviations of Δ_{PDF} and Δ_{CPDF} of all $+COS$ functions considered above for approximation of the normalized ($\sigma=1$) Gauss distribution are given together in Table 3 for comparison.

Each Normal distribution with arbitrary σ_i may be also similarly approximated by some COS^2 function of $A_i=B_i$ and $X_i=1/(2A_i)$ from columns 2–6 of Table 3. From (9a) results

$$A_i \sigma_i = A \sigma \quad \text{and} \quad X_i \sigma = X \sigma_i. \quad (10)$$

Three examples of COS^2 approximations for Gauss PDFs of $\sigma=(0.5, 1, 2)$ are given on Fig. 5.

4. SHORT NOTE ABOUT CONVOLUTIONS OF +COS FUNCTION

Result of convolution of two similar $+COS$ distributions $f(x)=A(1+\cos 2\pi xA)$ is shown on Fig 6a. For this example of $+COS$ function $A = B = 0.2$, $X = 2.5$, $\sigma_{+\cos}=0.904$ $\sigma_G = 0.997$. Excess=2,4 (as for triangular PDF). Convolution result is not $+COS$ function – it is nearer to Normal PDF.

Table 3. Parameters of few + COS distributions and of their differences Δ PDF, Δ CPDF to Gauss of $\sigma=1$

+COS $X_{APPROX}=X$	TYPE	Top point	$\sigma_{+COS}=1$	Green $f_{GR}(x)$	$\min(\Delta f)^2$	$\min \Delta f $	$\min(\Delta F)^2$	$\min \Delta F $
	Curve no	2	3	4	5	6	7	8
A	given value of A				optimal A		optimal A and B	
		0.200	0.181	0.159	0.186	0.196	0.178	0.179
B	B=A						0.220	0.219
$X=1/2B$		2.51	2.77	3.14	2.69	2.54	2.27	2.28
σ_{+COS}		0.906	1	1.136	0.971	0.922	0.937	0.936
Δ PDF $\Delta F = F_{+COS} - F_G$	$\min \Delta f$	-0.022	-0.037	-0.0806	-0.027	-0.020	-0.0015	-0.0013
	$\max \Delta f$	0.020	0.028	0.0446	0.024	0.020	0.012	0.012
	$\text{mean } \Delta f$	0.0024	0.001	0.0003	0.0013	0.0021	0.0050	0.0049
	$\overline{\Delta f}$	0.014	0.020	0.0389	0.016	0.0132	0.0043	0.0044
Δ CPDF $\Delta F = F_{+COS} - F_G$	$\min \Delta F$	-0.019	-0.018	-0.0483	-0.012	-0.016	-0.012	-0.011
	$\max \Delta F$	0.019	0.018	0.0483	0.012	0.016	0.012	0.011
	$\text{mean } \Delta F$	0.000	0.000	0.0000	0.000	0.000	0.000	0.000
	$\overline{\Delta F}$	0.012	0.010	0.0284	0.008	0.010	0.007	0.007

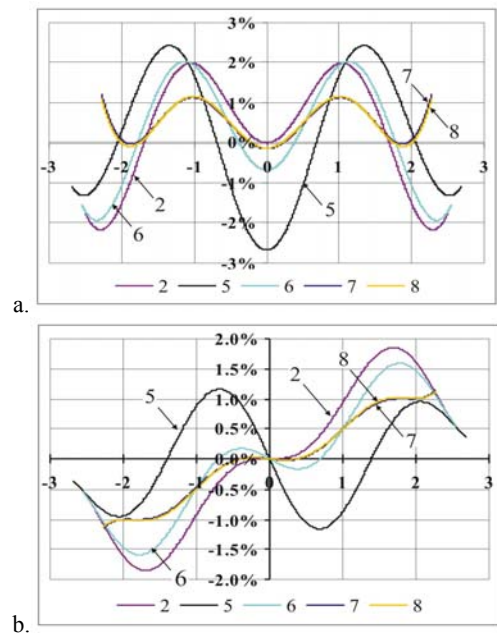
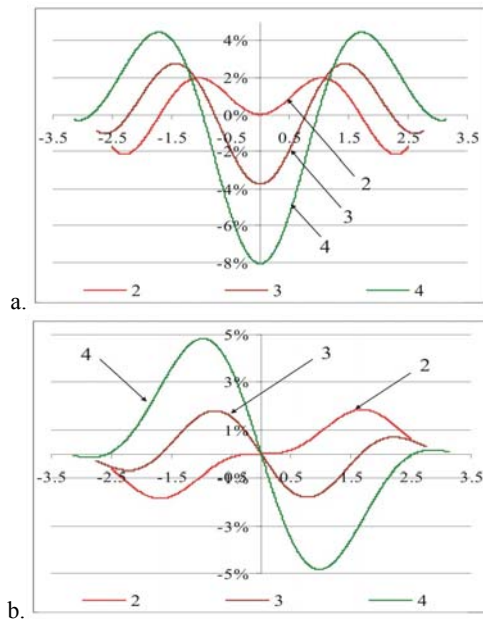


Fig. 2. Differences of COS^2 no 2, 3, 4 distributions and Normal distribution $N(0,1)$, i.e. of $\sigma=1$: a. of Δ PDF-s, b. of Δ CPDF-s

Fig. 3. Differences of few +COS distributions of Table 3 and Normal distribution $N(0,1)$: a. of Δ PDF-s, b. of Δ CPDF-s

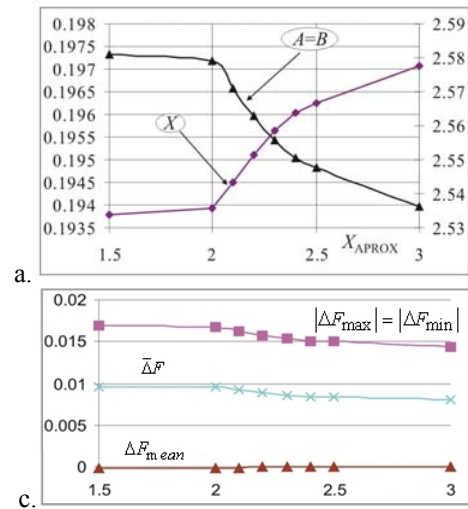
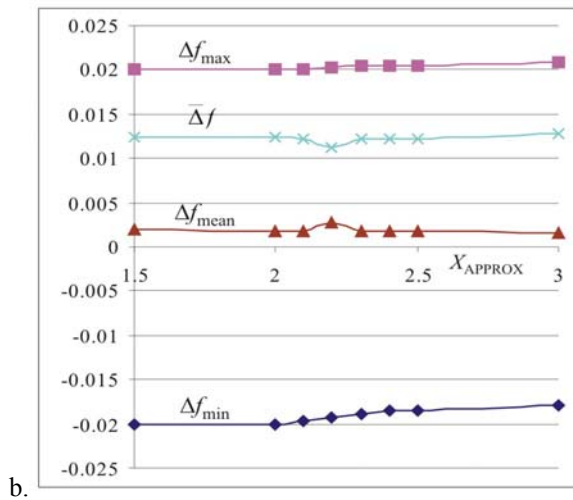


Fig. 4. Dependence of parameters on the range X_{APPROX} of approximation of the Normal distribution by COS^2 : a. of A and X, b, c. of accuracy parameters of Δ PDF and Δ CPDF if ranges X_{APPROX} have optimal A found by LSM method.

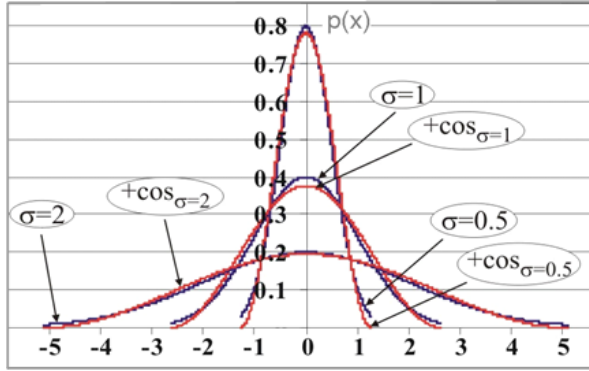


Fig. 5. Gauss PDFs of $\bar{x} = 0$ and $\sigma = (1/2, 1, 2)$ and their +COS approximates.

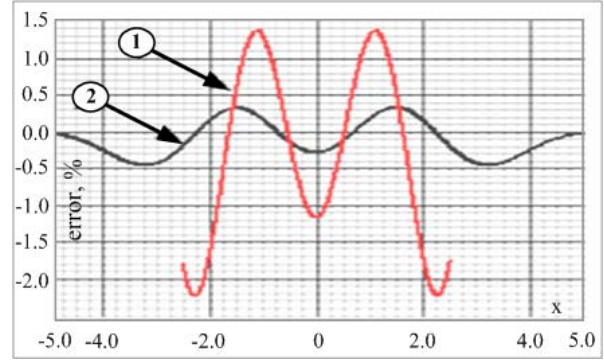


Fig. 6b. Differences Δ_{PDF} : of Gauss and COS^2 – curve 1, of their convolutions PDFs – curve 2.

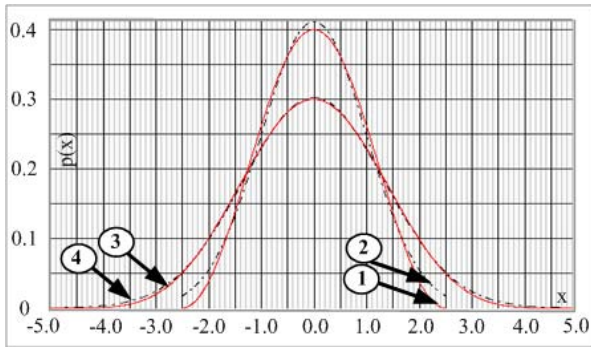


Fig. 6a. PDF of: 1 – COS^2 distribution and 3 – convolution of two identical COS^2 ; 2 – and 4 – their best fitting Gauss PDFs (dotted lines).

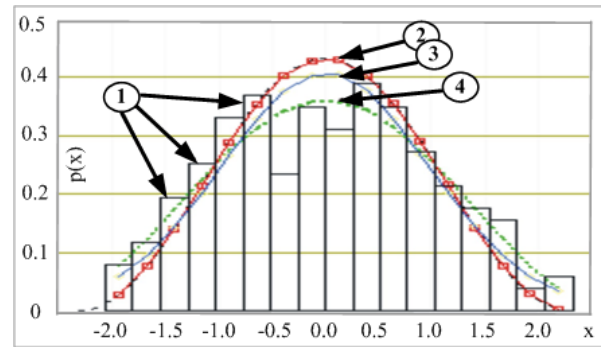


Fig. 7. 1 – Histogram h_1 of the sample of simulated data, 2 – sample Gauss PDF and two COS^2 PDF models: 3 – of the half-range X_1 to extreme x_n observation, 4 – of the range $2X_2$ obtain from SD of sample by (9a).

Standard deviation of convoluted two similar non correlated functions type +COS is

$$\sigma_{2(+COS)^*} = \sqrt{\sigma_{+COS}^2 + \sigma_{+COS}^2} = \sqrt{2} \sigma_{+COS}$$

Its SD $\sigma_{2(+COS)^*} = 1.28$. Excess = 2,71.

Full (\pm) range is $4X = 10$, $\sigma_{Best Gauss} = 1.33$.

Convolution of COS^2 with various PDF gives results very near to the same convolutions of Normal PDF. Differences of convolution function of two COS^2 PDF and its Gaussian function (Fig. 6b) are twice as smaller than those of single COS^2 , $\Delta_{PDF} = (0.0011, 0.0022)$.

5. EXAMPLE of UNCERTAINTY CALCULATIONS

Given is the sample of 200 values x_i of repeated measurement observations obtained by regular sampling of the simulated random population. Trend was removed from collected data similarly as in [3, 5]. Accuracy of their mean value as estimator has to be found. It will be done with application of Gauss and +COS models and then results should be compared.

Histogram of deviations from the sample mean value \bar{x} of 17 sub-ranges is shown on Fig. 7 as – 1.

There is also its Gauss PDF – 2. Sample range between extreme data x_{MIN} , x_{MAX} is 2.11. Compliance test χ^2 of COS^2 PDF of such range is not positive. So more wider two PDF-s are considered, i.e. 3 – of the half-range $X_1 = 2.31$ equal distance from 0 to extreme data x_{MAX} and 4 – of X_2 calculated from sample SD $S(x_i) = 0.978$ by formula (9a). Their extended uncertainties U of confidence levels k_p are in Table 4. Level of matching of Gauss and COS^2 PDF to histogram data are tested by SD of their differences – last two lines.

Table 4. Extended uncertainties of confidence levels k_p

Confidence level k_p (coverage factor)	$COS^2(0, X_i)$			$N(0, \bar{\sigma})$
	$X_1=1$	$X_1=2,31$	$X_2=2,71$	$S(x_i)=0.978$
	k_{PC}	Extended uncertainty $\pm U$		
0.500	0.265	0.043	0.051	0.047
0.683	0.385	0.063	0.074	0.069 (u_A)
0.900	0.596	0.097	0.114	0.114
0.950	0.683	0.112	0.131	0.136
0.990	0.816	0.133	0.156	0.178
0.997	0.878	0.143	0.168	0.205
1	1	0.163	0.192	∞
Criterion χ^2	$\chi_{14, 0.05}^2 = 23.7$	19,5	6,05	7,98
$u_A(U) = \sqrt{\sum (f(x_i) - h_{ii})^2} / \sqrt{n}$		0.071	0.012	0.016
$u_A(U) \%$		5,0%	6,4%	7,8%

Results of measurand $x = \bar{x} \pm U$ (where the extended uncertainty U of \bar{x} has probability P) are:

$$\text{Gauss model: } x = \bar{x} \pm k_{PG} \frac{S(x_i)}{\sqrt{n}} = \bar{x} \pm 0.205 \quad P = 0.997$$

$$\text{COS}^2 \text{ model: } x = \bar{x} \pm k_{PC} \frac{X_1}{\sqrt{n}} = \bar{x} \pm 0.143 \quad P = 0.997$$

$$\text{or } x = \bar{x} \pm k_{PC} \frac{X_2}{\sqrt{n}} = \bar{x} \pm 0.192 \quad P = 0.997$$

Extended uncertainty with confidence level 0,997 of the Gauss model is of 43.4 % or 22% higher than of the COS^2 of X_1 or X_2 half-ranges. The mean value \bar{x} in the case of COS^2 model is inside the range $\pm(2.36 \text{ or } 2.78)u_A$ of Gauss PDF with the probability nearly 1.

6. FINAL CONCLUSIONS

The unconventional PDF of cosine based function, of proposed symbol $+COS$ is considered. The suggested unconventional model of measurand probability density function is a shifted up (upward moved) one period of cosine function. This model is expressed by a well known, easy to generate trigonometric function.

An especially useful form of $+COS$ distribution is COS^2 when the amplitude A of cosine and shifting value B are equal. It means that the range of the random variable density has limits in points $\pm\pi$ in radians, where cosine is touching the horizontal axe. This is the two parameter model only as the Gauss distribution is and provides a good tool for evaluation of uncertainties. This function of properly chosen parameters approximates very well the central region of Gauss PDF in the range up to about $\pm 2,5$ standard deviation and gives better estimated uncertainty of the data of limited dispersion than unlimited Gauss PDF.

The COS^2 distribution does not have such inconvenience, which is characteristic for Gaussian distribution of which its left and right borders of tails are not limited up to $\pm \infty$. In practice it is impossible for empirical observations to get values so widely distributed even with very small probability.

The COS^2 distribution is very similar to Normal distribution and for the most cases the precision of fitting about $\pm 2\%$ in the range of its one full period is obtained. For more narrow ranges the better accuracy of approximation can be achieved.

The best fitting, even near $\pm 1\%$ of approximation of Gaussian distribution by $+COS$ PDF, is possible to achieve if cosine function is upward moved just a bit more than the value of cosine amplitude A . However, in such a case the cosine distribution is bounded similarly to a Normal distribution with cut off tails.

Distribution COS^2 has value of kurtosis nearly equal to triangular one, so it is also possible to use for it more accurate two-component estimator of PDF [8].

The convolutions of cosine based distributions approached Normal distribution; some operations are easier, use less time and memory consuming.

As the generation and simulations of cosine based distribution function are simpler, so they can be

implemented in single chip computers or in small micro-calculators.

The discussed cosine based distribution would be implemented in transducers and simpler instruments equipped with automatic uncertainty estimation to indicate results on line with required confidence [5, 6].

It may be also worth including the cosine based distribution to basic PDF distributions used in the routine data assessment.

The additional information that experimental data values of the sample are distributed in limited range allows to use also other non-Gaussian distributions of limited width as model of data histogram. Some of them are given in Table 1 of GUM Supplement 1 [2].

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