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# NOVEL AND ESTABLISHED CONCEPTS FOR CONSIDERING CORRELATION IN UNCERTAINTY EVALUATION

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**Abstract** – Modern uncertainty evaluation is based on both the knowledge about the measuring process and the input quantities contributing to the measurement result [1-2]. Very often, two or more of the input quantities are not independent from each other. The combined uncertainty can be enhanced or decreased by such correlation. In everyday practice, however, correlation is often ignored since the relevant uncertainty documents do not provide ready-for-use procedures for proper treatment of correlation. The paper provides practical techniques for identifying and quantifying correlation in measurements. Starting from a systematic modelling procedure [3-4], a concept is presented that allows to easily include correlation in the measurement model and to properly estimate correlation coefficients or correlated fractions of the related input quantities either from existing (statistical) data or from other (non-statistical, logical) knowledge [5-6]. Three possible ways to take correlation into consideration when evaluating measurement uncertainty are described and discussed.

**Keywords:** Measurement uncertainty, correlation, logical correlation

## 1. INTRODUCTION

The Bayesian concept underlying the GUM framework [1-2] is based on both the knowledge about the measuring process and the input quantities contributing to the combined uncertainty. Very often, two ore more of the input quantities are not independent from each other. Depending on the kind of influence of the correlated quantities, and on the measurement method employed, correlation can result in an increase or decrease of uncertainty: The combined uncertainty for the sum or the product of correlated quantities is enhanced and that for the difference or ratio is decreased, sometimes considerably [5-6].

Since there is a deplorable lack of comprehensible guidance on consistent and practicable treatment of correlation, in everyday practice, it is often simply ignored.

For practitioners, it is important to know in which cases correlation might be expected and where it does come from. Input quantities are often correlated because the same physical measurement standard, measuring instrument, reference datum, or even measurement method having a significant uncertainty are used for estimating their values [1]. Correlation can also exist if different standards are used which have been calibrated in the same experiment or even in the same laboratory. Furthermore, correlation can be expected if, with respect to the measuring procedure, significant systematic errors (including subjective ones) are to be considered.

On the other hand, for example, in case of doing difference measurement, correlation can be a highly desired effect, such as, in comparing weights, electrical resistances etc. by means of comparators [5-6]. In these cases, correlation results in a decrease of uncertainty.

The paper provides practical techniques for identifying and quantifying correlation in measurements. A concept is presented that allows to easily include correlation in the measurement model and to properly estimate correlation coefficients or correlated fractions of the related input quantities either from existing (statistical) data or from other (non-statistical, logical) knowledge [5-6].

In measurement practice, it is clearly to be distinguished between „logical“ (non-statistical) and statistical correlation. The last-mentioned kind of knowledge appears in situations quantities are repeatedly measured. Generally, correlation is caused by a dependence of two or more quantities on one or more common („third“) quantities. The paper describes three possible ways to take correlation into consideration when evaluating the measurement uncertainty: Resolving correlation by introducing known dependencies on another (third) quantity, the „classical way“ of Gaussian uncertainty propagation and introducing so-called auxiliary quantities representing the „correlated fraction“ of two or more quantities.

## 2. CONSIDERATION OF CORRELATED QUANTITIES

For the standard GUM procedure see [1], for the concept of modelling see [3-4].

### *2.1. Expectation, variance and covariance*

The expectation value of an input quantity $X_i$ is the best estimate of the value of that input quantity,

$$x_i = \mathrm{E}\left[X_i\right] = \int_{-\infty}^{+\infty} g_{Xi}\left(\xi_i\right)\xi_i d\xi_i\,, \qquad (1)$$

and its standard deviation is the uncertainty $u_{xi}$ associated with this estimate,

$$u_{xi} = \sqrt{\mathrm{Var}\left[X_i\right]} = \left\{ \int_{-\infty}^{+\infty} g_{Xi}\left(\xi_i\right)\left(\xi_i - x_i\right)^2 d\xi_i \right\}^{1/2} . \qquad (2)$$

The covariance of two (random) variables $X_i$ and $X_j$ is a measure of their mutual dependence [1]. It is defined by

$$\mathrm{Cov}\left[X_i, X_j\right] = \mathrm{E}\left[\left(X_i - x_i\right)\left(X_j - x_j\right)\right] = u_{xixj} . \qquad (3)$$

The covariance can also be expressed by the product of individual variances $u_{xi}$ and $u_{xj}$ and the correlation coefficient

$$r_{ij} : \; u_{xixj} = u_{xi} \cdot u_{xj} \cdot r_{ij} . \qquad (4)$$

The correlation coefficient characterizes the strength of interdependence of the quantities $X_i$ and $X_j$. Its value is *zero* if these quantities are completely independent of each other and it tends to be ($\pm$) *unity* if there is an unambiguous functional relationship between them. A detailed interpretation of the meaning of the correlation coefficient with the example of a linear regression of two quantities $X_i$ and $X_j$ from observed data is given in [6].

### *2.2. Correlated quantities in modelling*

When carrying out modelling, it is an indispensable prerequisite that the input quantities being correlated really appear in the cause-and-effect relationship and in the model equation respectively [3-4].

If, for example, a standard weight *W*s is established by two individual (patched) weight pieces SRC1 and SRC2, this is clearly to be represented in the cause-effect relationship (see Fig. 1). In case of this example shown in Fig. 1, due to their calibration within the same experiment and the same laboratory respectively, the (unknown) errors of the patched standards used, $\delta W_{\mathrm{S1}}$ *and* $\delta W_{S2}$, and, thus, their weighing values $W_{\mathrm{S1}}$ and $W_{\mathrm{S2}}$ can be assumed to be correlated quantities. This correlation might be caused by using the same standard (of higher order) and the same weighing apparatus for their calibration.

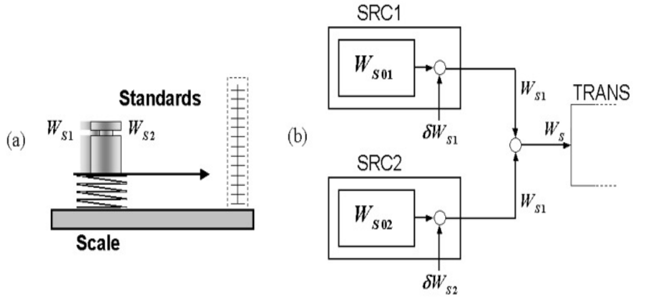


Fig. 1. Example: Depiction of using „patched“ standards for the calibration of a scale, (a) illustration, (b) cut-out of the respective cause-effect relationship. Symbols see text

Therefore, the mathematical cause-effect relationship for the patched standard must always read

$$W_{\mathrm{S}} = W_{\mathrm{S01}} + W_{\mathrm{S02}} - \delta W_{\mathrm{S1}} - \delta W_{\mathrm{S2}} , \qquad (5)$$

where $W_{\mathrm{S01}}$ and $W_{\mathrm{S02}}$ are the nominal weighing values of the weights and $\delta W_{\mathrm{S1}}$ *and* $\delta W_{S2}$ are the respective errors.

Another example, taken from the GUM [1], is given with Fig. 2: The partial resistance of a resistance decade that are considered to be correlated are to be treated as separate elements $R_1, \ldots, R_{10}$ to be associated with individual expectations and uncertainties. Therefore, the combined resistance $R_{\mathrm{DEC}}$ in the mathematically expressed cause-and-effect relationship must be written $R_{\mathrm{DEC}} = R_1 + R_2 + \ldots + R_{10}$.

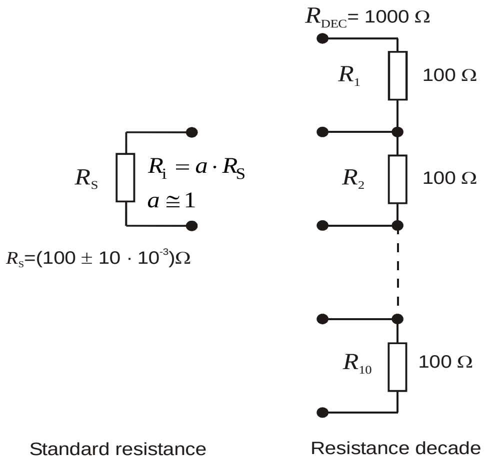


Fig. 2. Illustration of the calibration of a resistance decade with a standard resistance (see GUM [1] 5.2.2 and F.1.2.3)

Wrongfully, often one uses $R_{\mathrm{DEC}} = 10 \cdot R_1$ that corresponds with a correlation coefficient $r = 1$. The example shows that for the inclusion of correlated quantities, the same modelling components and the same concept for establishing the measuring chain as for uncorrelated measurements can be utilized [3-4]. But it is an absolutely indispensable prerequisite that the correlated quantities do appear in the model equation.

## 3. ORIGIN OF CORRELATION

For practitioners it is important to know where correlation might be expected and where it does come from. Correlation is present in many measurements and, dependent on the model for the evaluation of uncertainty, it enhances or decreases the combined uncertainty.

In practice, input quantities are often correlated because the same physical measurement standard, measuring instrument, reference datum, or even measurement method having a significant uncertainty is used in the estimation of their values [1]. This may be comprehensible illustrated with the example of the determination of a rectangular area by measuring the length and the width by means of the same measuring instrument.

The above statement applies also if different standards are used but these standards have been calibrated in the same experiment or even in the same laboratory (see the example depicted in Fig. 1).

Furthermore, correlation can be expected if, with respect to the measuring procedure, significant systematic errors (including subjective ones) are to be considered.

On the other hand, correlation is a highly desired effect in case of doing any difference measurement, such as, for example, comparing weights, electrical resistances etc. by means of a comparator [6]. In these cases, correlation results in a decrease of uncertainty.

In measurement practice, is clearly to be distinguished between „logical“ (non-statistical) and statistical correlation. The latter appears in situations where quantities are repeatedly measured (see 4.1) and the former is caused by a dependence of two or more quantities on one or more common („third“) quantities.

## 4. MATHEMATICAL TREATMENT OF CORRELATION

### *4.1. Statistical correlation*

Both, statistical and logical correlations are based on systematic effects. For mathematical treatment, the so-called statistical correlation is the simplest case. It is related to repeated measurements (observations); e.g. of the quantities $X_i$ and $X_j$. Then, the covariance associated with the expectation values can be calculated by

$$u_{xixj} = \frac{1}{n(n-1)} \sum_{k=1}^{n} \left( q_{ik} - \overline{q}_i \right)\left( q_{jk} - \overline{q}_j \right) \,, \qquad (6)$$

where $q_{ik}$ and $q_{jk}$ are the individual observations of the quantities $X_i$ and $X_j$, and $\overline{q} = \frac{1}{n}\sum_{k=1}^{n} q_k$. The correlation coefficient, $r_{ij}$, can easily be derived from the above estimated covariance by $r_{ij} = \frac{u_{xixj}}{u_{xi} \cdot u_{xj}}$ [1]. (7)

In accordance with the ISO-GUM procedure [1], correlation is to be taken into account by Gaussian uncertainty propagation.

To demonstrate this, we study two simple models:

$$Y_+ = X_i + X_j \text{ and } Y_- = X_i - X_j \,. \qquad (8)$$

Assume, we generate $n$ samples of possible values $\left( q_{ik}, q_{jk} \right)$ and, using standard statistical procedures, compute the mean values, the experimental standard deviations of the means and the covariance associated with the mean values. For the expectation values we obtain

$$y_\pm = \bar{x}_i \pm \bar{x}_j \,. \qquad (9)$$

Gaussian uncertainty propagation yields

$$u_{y+}^2 = u_{xi}^2 + u_{xj}^2 + 2u_{xixj} \text{ and } u_{y-}^2 = u_{xi}^{-2} + u_{xj}^2 - 2u_{xixj} \qquad (10)$$

respectively.

### *4.2. “Logical” (non-statistical) correlation*

“Logical“ correlation is always caused by the dependence of two or more quantities on one (or more) common (third) quantity. For formal description we consider that at least two given quantities $X_i$ *and* $X_j$ may be influenced by a set of $n$ quantities $\mathbf{Q}_L = (Q_1, ..., Q_l, ..., Q_m)^{\mathrm{T}}$.

For instance, consider the determination of the area of a rectangle $A_r$ by measuring the height $H$ and width $W$ by means of the same instrument. The simplest model would be given by

$$A_r = H \cdot W \,. \qquad (11)$$

Assume the model functions (of the type (9)) for the input quantities $H$ and $W$ are:

$$H = f_H\left( h, \delta H, \Delta M_H \right) = h + \delta H + \Delta M_H \text{ and}$$
$$W = f_W\left( w, \delta W, \Delta M_W \right) = w + \delta W + \Delta M_W \,, \qquad (12)$$

where $h$ and $w$ are the values indicated by the instrument used, $\delta H$ and $\delta W$ are quantities that account for various (random) effects and imperfections, in particular for the finite resolution of the instrument, and $\Delta M$ is the quantity that accounts for the instrumental error of the instrument used and, therefore, for a joint uncertainty contribution of the quantities *H* and *W*. The model for the measurand $A_r$ would then be given by:

$$A_r = \left( h + \delta h + \Delta M_H \right)\left( w + \delta W + \Delta M_W \right) \,. \qquad (13)$$

In this case of „logical“ correlation of type-B evaluated quantities, the “statistical” Equations (6) and (7) cannot be used to determine the resulting correlation coefficient. Nevertheless, the GUM [1] uses the covariance and the correlation coefficient for calculating the measurement uncertainty.

The quantities, $X_i$ and $X_j$ that depend on the same set of quantities $Q_1, ..., Q_l, ..., Q_m$ have the expectations $q_l = \mathrm{E}\left[ Q_l \right]$. Therefore, the covariance associated with the expectations $x_i$ and $x_j$ can be estimated by [1]

$$u_{xixj} = \sum_{l=1}^{m} \frac{\partial f_i}{\partial q_l} \cdot \frac{\partial f_j}{\partial q_l} \cdot u_{ql}^2 \; , \tag{14}$$

where $X_i = f_i\left(Q_1,...,Q_m\right)$ and $X_j = f_j\left(Q_1,...,Q_m\right)$.

### *4.3. Correlation in uncertainty calculation*

Generally, there are three possible ways to take correlation into consideration when evaluating the combined measurement uncertainty:

a. If the relationship between the correlated quantities can unambiguously be expressed, e.g. in case of knowing their functional dependencies on another (third) quantity, this relationship should be introduced in the model equation. This way will result in resolving the correlation. If possible, this way is to be preferred.

b. If the correlation coefficient(s) or the respective covariance(s) is/are sufficiently well known, they can be taken into account and propagated as recommended by the GUM [1], i.e. using the Gaussian uncertainty propagation.

c. Since correlation is always related to systematic effects, one can formally introduce an auxiliary quantity that „represents“ the correlated fraction of the quantity. In some cases, if the physics of the measurement is well known, this allows to resolve correlation straightforward by modelling explicitly the dependence of two or more input quantities on the same systematic effect expressed by the auxiliary quantity [5-6].

***Way a:***

Starting with the example given above (see 3.2), the easiest way to calculate the measurement uncertainty is to directly utilize the model equation (13).

For greater ease in writing, we introduce $\delta H_h = \delta H/h, \delta W_w = \delta W/w$, and analogously $\Delta M_H$ and $\Delta M_W$. We know that the quantities $H$ and $W$ (see example above) are correlated due to their common dependence on the measurement error of the instrument used. The model for the area $A_r$ may then be written as

$$\begin{aligned} A_r &= h \cdot w\left(1+\delta H_h + \Delta M_H + \delta W_w + \Delta M_W\right) \\ &\quad + h \cdot w\left(\delta H_h\left(\delta W_w + \Delta M_W\right) + \left(\delta W_w + \Delta M_W\right)\right), \end{aligned} \tag{15}$$

and the best estimate for the value of $A_r$ is its expectation. We assume that $\mathrm{E}\left[\delta H\right] = \mathrm{E}\left[\delta W\right] = \mathrm{E}\left[\Delta M\right] = 0$. This leads to

$$\begin{aligned} a_r &= \mathrm{E}\left[A_r\right] = hw\left(1+\mathrm{E}\left[\Delta M_H \cdot \Delta M_W\right]\right) \\ &= hw + u^2\left(\mathrm{E}\left[\Delta M\right]\right) \end{aligned} \tag{16}$$

The term $u_{\Delta M}^2 = u^2\left(\mathrm{E}\left[\Delta M\right]\right)$ would not appear if one uses a linear model. Therefore, this term can usually be neglected. Assuming this, the linearized model is obtained:

$$A_r = h \cdot w + w\delta H + h\delta W + \Delta M\left(h+w\right) \; . \tag{17}$$

From this model, one can compute the uncertainty as

$$u_{ar}^2 = w^2 u_{\delta H}^2 \cdot h^2 \cdot u_{\delta W}^2 + u_{\delta M}^2 \cdot \left(h+w\right)^2 \; , \tag{18}$$

where for the ease in writing the same symbol is used for the quantity and for its value.

***Way b:***

In practice one is often not given a detailed uncertainty budget for the input quantity but only a combined uncertainty and an estimated correlation coefficient and the correlation coefficient for any two or more measurements is $r_{l1l2}$. By inspection of the relationships in (11) one may infer that

$u_{l1}^2 = u_{\delta H}^2 + u_{\Delta M}^2 = u_{l2}^2 = u_{\delta W}^2 + u_{\Delta M}^2$, and

$$r_{l1l2} = \frac{\mathrm{Cov}\left[L_1, L_2\right]}{u_{l1} \cdot u_{l2}} = \frac{u_{\Delta M}^2}{u_{l1} \cdot u_{l2}} \Rightarrow u_{\Delta M}^2 = r_{l1l2} \cdot u_l^2 \; . \tag{19}$$

The uncertainty associated with the area could then be computed by means of the well-known formula

$$u_{ar}^2 = l_2^2 u_{l1}^2 + l_1^2 u_{l2}^2 + 2 l_2 u_{l1} \cdot u_{l2} l_1 \cdot r_{l1l2} . \tag{20}$$

Because of $u_{l1} = u_{l2}$ (see example above) one obtains:

$$u_{ar} = u_l \cdot \sqrt{l_1^2 + l_2^2 + 2 \cdot l_1 l_2 \cdot r_{l1l2}} \; . \tag{21}$$

This is fairly simple and can be used in corresponding cases. However, in case of more complex models, this simple approach may fail.

***Way c:***

The way is to introduce an auxiliary quantity $X_{\mathrm{C}}$ that is presumed to cause correlation. For reasons that so will be evident, we represent now any quantity $X$ by

$$X = x + u_x \cdot X_{\mathrm{Std}} \tag{22}$$

where $x = \mathrm{E}\left[X\right]$ and $\mathrm{Var}\left[X_{\mathrm{Std}}\right] = \mathrm{E}\left[\left(X_{\mathrm{Std}}\right)^2\right] = 1$ [6].

Using this form, one can represent the correlated quantities, e.g. the diameter $D$ and the height $H$ of a given cylinder, as a sum or difference of an uncorrelated and an auxiliary quantity that accounts for correlation:

$$D = d + u_{\mathrm{du}} \cdot D_{\mathrm{u,Std}} + u_{xc} \cdot X_{\mathrm{c,Std}} \text{, and} \tag{23}$$

$$H = h + u_{\mathrm{h,u}} \cdot H_{\mathrm{u,Std}} \oplus u_{xc} \cdot X_{\mathrm{c,Std}} \qquad (24)$$

where $\oplus$ is to be derived separately [6]. This representation must yield the correct uncertainties. Therefore, we additionally request that

$$u_{d\mathrm{u}}^2 = u_d^2 - u_{x\mathrm{c}}^2, \text{ and } u_{h\mathrm{u}}^2 = u_h^2 - u_{xc}^2. \qquad (25)$$

Furthermore, this representation must yield the correct covariance or correlation coefficient. Therefore,

$$\mathrm{Cov}\left[D, H\right] = u_d \cdot u_h \cdot r_{d,h} = +u_{\mathrm{c}}^2 \qquad (26)$$
$$\Rightarrow u_{\mathrm{xc}}^2 = u_d \left(\oplus r_{d,h} \cdot u_h\right).$$

For greater ease in writing we use a simple substitution: $\sin\alpha = \sqrt{|r|}$. The models function for the volume of the cylinder $V_{\mathrm{cyl}}$. is now given by:

$$V_{\mathrm{cyl}} = \pi\left(d + u\cos\alpha\, D_{\mathrm{u,Std}} + u\sin\alpha\, X_{\mathrm{c,Std}}\right)^2 \left(h + u\cos\alpha\, H_{\mathrm{u,Std}} + u\sin\alpha\, X_{\mathrm{c,Std}}\right), \qquad (27)$$

Only a few terms remain when taking the expectation:

$$v_{\mathrm{cyl}} = \pi\left(d^2 h + u^2\left(h + 2\sin^2\alpha\left(d+1\right)\right)\right), \qquad (28)$$

and uncertainty respectively:

$$u_{V\mathrm{cyl}}^2 = \pi^2 u^2\left(4h^2 d^2 + d^4 + 2\sin^2\alpha\, hd^3\right). \qquad (29)$$

If a linear model is not sufficient, Monte-Carlo techniques can be utilized. As an often used alternative to the above approach one might also consider the *Cholesky* factorisation of the uncertainty matrix $\boldsymbol{U_x}$ [6].

## 5. CONCLUSION

This paper identifies typical measurement situations in which correlation may be of importance. Furthermore, based on a modelling procedure, the paper presents three possible ways to take correlation into consideration when evaluating the measurement uncertainty. Primarily, these are the two „classical ways“ resolving the correlation by Gaussian uncertainty propagation. The calculation of correlation coefficients from statistical data as well as their estimation in case of „logical“ correlation of type-B evaluated quantities is explained. A third way to take correlation into consideration can be done by introducing a so-called auxiliary quantity. Important advantages of this method are that it does not need any sensitivity coefficients and – in connection with Monte-Carlo techniques – it simplifies the treatment of complex problems significantly.

Systematic inclusion of correlation in the modelling and uncertainty-evaluating procedure may be considered as an important improvement of modern uncertainty evaluation.

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