XIX IMEKO World Congress
Fundamental and Applied Metrology
September 6–11, 2009, Lisbon, Portugal

DIGITAL NOTCH FILTERS IMPLEMENTATION WITH FIXED-POINT ARITHMETIC

Eduardo Pinheiro, *Octavian Postolache*, *Pedro Girão*

Instituto de Telecomunicações, Lisboa, Portugal, eduardo.pinheiro@ist.utl.pt, poctav@alfa.ist.utl.pt, psgirao@ist.utl.pt

Abstract – Many studies have been developed aiming to improve digital filters realizations, recurring to intricate structures and analysing the error´s behaviour. The work presented in this paper analyses the feasibility of fixed-point implementation of classical IIR notch filters (Butterworth, Chebyshev, Bessel and elliptic), and also the effect of the quality factor and normalized cut-off frequency in the number of significant bits necessary to represent the coefficients, to scrutinize the deformations the filters suffer for distinct design specifications.

The work focuses especially in the implementation of power line notch filters used to improve the signal-to-noise ratio in biomedical signals. The obtained results, when quantizing the digital notch filters, show that by applying second order sections decomposition, low-order digital filters may be designed using only part of double precision capabilities, while high-order notch filters with harsh design constrains are implementable using double precision, but only in second-order sections. Thus, it is shown that to optimize computation time in real-time applications, an optimal digital notch filter implementation platform should have variable arithmetic precision.

Keywords: digital filter implementation, digital filter word length effects, notch digital filters.

1. INTRODUCTION

Notch filters are very important in a wide variety of instrumentation applications, from telecommunications to biomedical signals processing, where often it is necessary to remove a narrow band or even a single frequency of the measurement signal. Digital implementation of these filters is preferable to an analog implementation due to drift absence and straightforward design of higher quality factors. Nevertheless, digital filters' implementation has accuracy limitations due to the arithmetic's finite precision [1-4], an issue that is much more significant in fixed-point arithmetic than in floating-point.

Due to the ease of designing and calculating the coefficients of high-performance digital IIR filters, the filter outcome is taken for granted, but, particularly if dealing with limited capacity fixed-point platforms (such as microcontrollers, digital signal processors, and field-programmable gate arrays) or with very demanding design constraints, the filtering stage may have a pernicious effect on the signal, completely missing its purpose.

This problem has been studied [2-9] and, disregarding additional error sources originated from the A/D and D/A conversions, the key issues are:

I. Quantization of the input signal into a finite set of discrete levels;
II. Representation of the filter's coefficients by a short number of bits;
III. Propagation of rounding errors occurred in arithmetic operations.

To evaluate these errors influence in the final filter output, several approaches have been proposed [2-9,12-14]. If errors type-I are assumed to be random variables with a uniform probability distribution, a number of analysis tools is available to characterize their behaviour [10-14]. Errors type-III are incessantly subject of reductions through the implementation of novel structure variations [1-2,5,15-17] based in state-space structures and direct form I with error feedback, also known as noise shaping or error spectrum shaping [5,9,18].

Type-II errors also have comprehensive bibliography reporting studies on important implementation issues; some instability thresholds due to these errors were derived [6,19], not including notch filters, coefficients sensitivity approach [15,18-20], and structural changes to minimize the impact of these errors [2,5,8,17].

Considering specifically biomedical applications, some studies have analysed the digital filters distortion effect on the signal [15], but the feasibility and the outcome of the implementation has not been discussed. Moreover, several biomedical studies ignore, to some extent, the higher-frequency components of the signals, implementing low-pass filters, or wide band-stop filters. Ballistocardiogram, electrocardiogram, electroretinogram, which have sampling frequencies from 200 Hz to 2 kHz, and other biomedical signals high-resolution processing systems benefit from the usage of power line notch filters.

Since acquisition systems work at distinct sampling rates, the analysis of digital notch filters performance at different normalized cut-off frequencies allows ensuring that most biomedical signals fit in the tested range, and so the conclusions are applicable to a broad variety of digital biomedical signal processing systems. Subsequently, MATLAB processing capabilities are used to evaluate the fixed-point arithmetic numerical accuracy requirements to realize several types of IIR notch filters, at different design specifications.

2. SECOND ORDER FILTERS

Using dedicated filter design software, floating-point double precision coefficients were computed for the following filter types: Butterworth, Chebyshev types I and II, and Elliptic. It was considered a normalized notch frequency vector Ω_0 with 9 points per decade spaced from 10^{-4} to 0.3 (totalling 30 points) and a quality factor vector Q also with 9 points per decade spaced from 1 to 10^4 (totalling 37 points) and filters of even orders from second to tenth were designed.

In view of the fact that the quantization induces pole movement, a stable filter after quantization may become unstable or even if the quantized filter is confirmed to be stable, its outcome may be unacceptable, thus stating that although the poles remain in the interior of the unit circle, the quantization is too coarse and the poles and zeros movement deforms the filter behaviour.

To diminish the poles and zeros wandering one valuable method is the implementation of the filter in second order sections (decomposing an N^{th} order filter in the product of N/2 second order filters), considering that the coefficients' quantization causes minor pole movement than in higher order sections. The impact of this option will also be evaluated.

2.1. Filters' definitions

The normalized frequency Ω is defined as the ratio between the frequency and the Nyquist rate, thus resulting in units of cycles per sample.

The quality factor Q is the ratio between Ω_0 and the bandwidth (difference between upper and lower cut-off frequencies Ω_1 and Ω_2), while the notch frequency Ω_0, the centre of the stop band, is the geometric mean of Ω_1 and Ω_2. Since results should be parameterized as functions of Ω_0 and Q and filter design algorithms process Ω_1 and Ω_2, (1) was used to obtain Ω_1 and Ω_2 from design specifications in Ω_0 and Q.

$$\left\{\begin{aligned} \Omega_0 &= \sqrt{\Omega_1\Omega_2} \\ Q &= \frac{\Omega_0}{\Omega_2-\Omega_1} \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} \Omega_2 &= \frac{\Omega_0}{2Q}\left(1+\sqrt{1+4Q^2}\right) \\ \Omega_1 &= \frac{{\Omega_0}^2}{\Omega_2} \end{aligned}\right\} \tag{1}$$

The filters were implemented using Direct-Form II, see Fig. 1, of the filter's transfer function H(z), represented in (2) for the second order case.

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}} \tag{2}$$

Stability assessment was made searching for poles of the filter's transfer function, *H(z)*, outside the unit circle. The *n* bit fixed-point filter's deviations to the floating point double precision format design (16 decimal digits of precision in calculations, IEEE decimal64 format) [21] was measured making use of it's frequency response magnitude, $|H_{n\ bit}(j\Omega)|$, root mean square error $\varepsilon_{n\,bit}$ (in dB), using (3),

$$\varepsilon_{n\,bit} = \sqrt{\sum_{\Omega=\Omega_{min}}^{\Omega_{max}} \left[\left|H_{n\,bit}(j\Omega)\right|_{dB} - \left|H_{float}(j\Omega)\right|_{dB} \right]^2} \tag{3}$$

where Ω is the test-frequency vector, $H_{n\_bit}(j\Omega)$ and $H_{floatt}(j\Omega)$ the transfer functions of both filters.

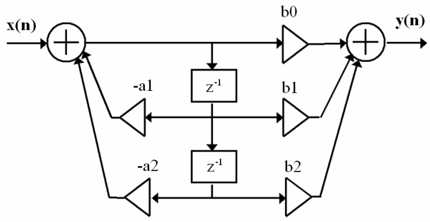

Fig. 1. Implementation of (2) in Direct-Form II.

It should be noticed that $\varepsilon_{n\ bit}$ could have been defined in linear units or using the phase or group delay difference, but since the magnitude in dB is the most widely employed method to assess filter response, the parameter $\varepsilon_{n\ bit}$ was chosen to measure directly this dissimilarity in dB. Filter deviations are problematic both in pass and in stop band, since deviations start to manifest in the stop band and afterwards spread to the pass band also, the metric (3) equally weights all frequencies.

3. SECOND ORDER FILTERS RESULTS AND DICUSSION

3.1. Filters' stability

Second order band-stop filters of the stated types were implemented using the defined Q and Ω_0 vectors. It was found that, for every filter type, only the 16 bit implementation was stable for all (Q,Ω_0) pairs and that the minimum quality factor to design an unstable filter, Q^{u_min} is 40, in the normalized notch frequency, $\Omega_0^{u\_min}$ was found 8×10^{-3}.

The (Q,Ω_0) pairs that generate unstable filters vary their position according to the number of bits of the implementation but not with the filter type.

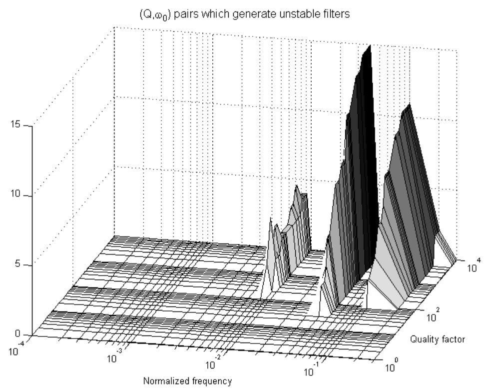

Fig. 2. Number of second-order unstable filters for each (Q,Ω_0) pair, in a total of 28 designed per (Q,Ω_0) pair.

Fig. 2 shows the number of unstable filters obtained for each (Q,Ω_0) pair (total of 1110), considering 10 to 16 bits implementations of the four filter types (28 designed per pair).

Regarding power line notch filters implementation in biomedical systems, the range of the normalized notch frequencies where the filter is unstable represents an important drawback because implementations with sampling rates from 2 kHz down to 200 Hz will cross the two main instability peaks found in Fig. 2. Despite this, if quality factors bellow 40 are tolerable, the implementation of 10 to 16 bit fixed-point IIR notch filters is straightforward.

3.2. Filters' deviations

Second order band-stop filters of the stated types were implemented using the defined Ω_0 and Q vectors. The results obtained for $\varepsilon_{n\ bit}$ in a second-order Butterworth filter, at a fixed Ω_0 of 0.05, thus situated in the more disturbing zone, with n from 10 to 16 bits, and variable Q are presented in Fig. 3.

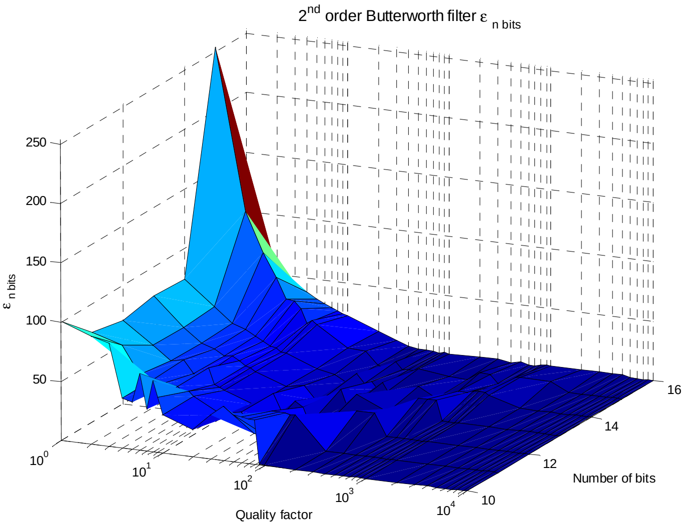

Fig. 3. Second-order Butterworth filter $\varepsilon_{n\ bit}$, with Ω_0 of 0.05, n from 10 to 16 bits, and 37 points Q vector.

Smaller values of the quality factor, namely 1, have the higher differences, which is due to the fact that the floating-point filter fixed-point implementation creates a deeper notch than the small deviation due to the fixed-point conversion is able to mimic truthfully. If Q is above 400, the filters have very small notch frequency attenuation and a small amplitude resonance peak both in fixed and floating-point implementations. For Q values above 1000 this peak vanishes and the filter acts as an all pass filter, having no discrepancy from fixed to floating point. To exemplify this behaviour in Fig. 4 it is plotted the magnitude of the 15 bit Butterworth filter frequency response, $|H_{15\ bit}(j\Omega)|_{dB}$, for variable Q.

Butterworth filters are presented as examples in the last two figures, but the other filters have exactly the same characteristics regarding the error and the magnitude response progress with the quality factor and all others also generate resonance peaks at very high quality factors.

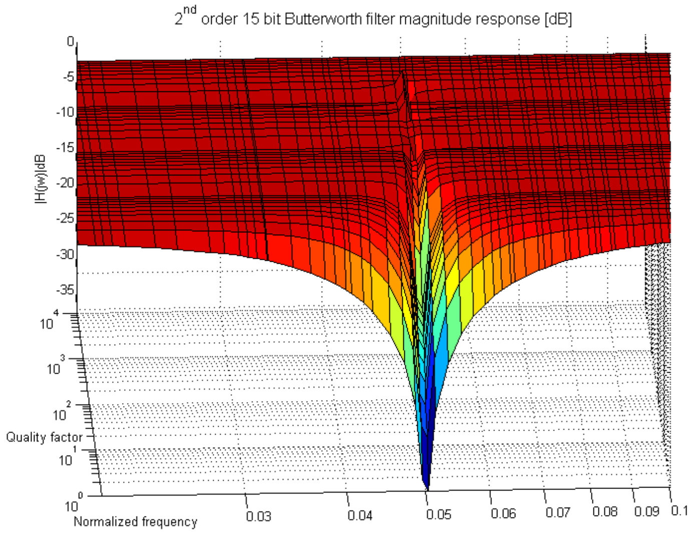

Fig. 4. Magnitude, in dB, of the 14 bit 2nd order Butterworth filter, $|H_{15\ bit}(j\Omega)|_{dB}$, with Ω_0 of 0.05 and the defined Q vector.

3.3. Filters' optimization

In these implementations we searched the minimum coefficient word length that guaranteed stability and the optimal word length, considering the defined accuracy metric. The filter demanding wider coefficients' word lengths to guarantee stability was the elliptic filter, and the Chebyshev type I was the most demanding to minimize error. The coefficients' word length dependency on Q and Ω_0 in these two cases are shown in Figs. 5 and 6.

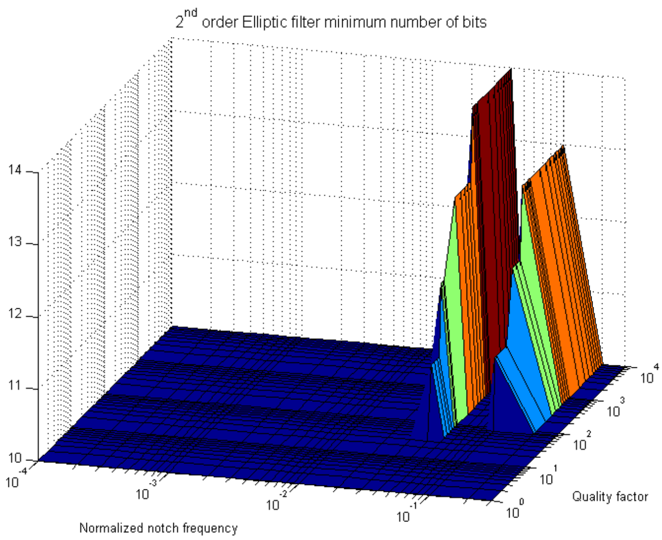

Fig. 5. Minimum coefficients word length to implement a stable second-order elliptic filter for the defined Q and Ω_0 vectors.

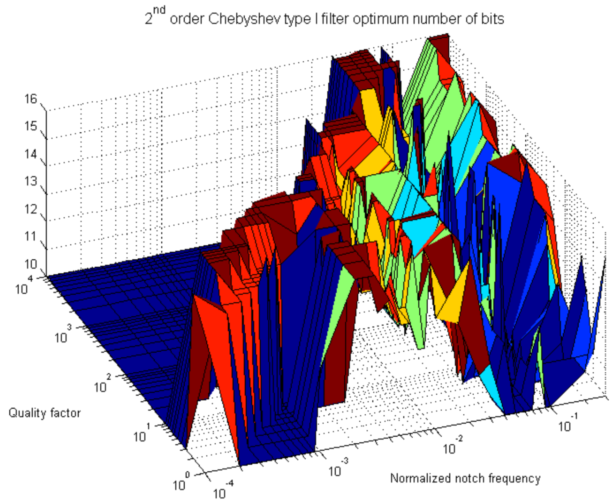

Fig. 6. Optimal coefficients' word length to implement a stable second-order Chebyshev type I filter for the defined Q and Ω_0 vectors minimizing the norm (3).

4. HIGHER ORDER FILTERS RESULTS AND DICUSSION

Repeating the design procedure, 4th, 6th, 8th and 10th order filters were implemented in single section and second-order sections. The results regarding filter's stability and quantization effects are subsequently presented.

4.1. Filters' stability

Regarding the filters' stability, Table 1 presents the number of stable filters designed for each order and each filter type, when using single section (SS) and second-order sections (SOS) implementations. The maximum coefficients' word length allowed was of 16 bits and the total number of filters for all (Q,Ω_0) pairs is 1110.

Table 1. Number of stable filters of 4th, 6th, 8th and 10th order.

Order	Type							
	SS				SOS			
	B	C1	C2	E	B	C1	C2	E
4	134	138	162	133	1110	1110	1110	1110
6	16	19	21	16	1110	1110	1110	1110
8	6	7	6	4	1110	1110	1110	1110
10	2	3	2	1	1110	1110	1110	1105

In this table it is visible that for the 4th order, the single section implementation is no longer valid, since only 12 to 14.6% of the filters implemented using this structure are stable. For even higher orders few designed filters implementations are characterized by stability.

When decomposing the filter structure into second-order sections implementation, the rearrangement of the coefficients allows the minimization of deviations from the poles real value in such a way that only five elliptic filters of 10th order are unstable.

Other important result is the minimum and the average number of bits required to ensure that the SOS filters are stable for all (Q,Ω_0). The results are presented in Table 2.

The second-order section filters preserve the behaviour presented in Fig. 2. Only a few tenths of them require more than 10 bits. When increasing the order the requirements of this residual minority also increase, but only 10 bits are needed to almost every filter implementation.

Table 2. Number of bits (average and maximum) to design stable filters for all (Q,Ω_0) pairs.

Order	Type							
	Average for all (Q,Ω_0)				Max for all (Q,Ω_0)			
	B	C1	C2	E	B	C1	C2	E
4	10.1090	10.1261	10.0919	10.1297	14	14	14	14
6	10.1270	10.1550	10.1243	10.1667	14	14	14	14
8	10.1360	10.1721	10.1468	10.2000	14	15	14	16
10	10.1505	10.1820	10.1595	---	14	15	14	>16

4.2. SOS Filters' deviations

The results of previous section 2.1 indicate the importance of analyzing not only the global (Q,Ω_0) mesh but also the zones with more demanding coefficients' word length to ensure stability. Table 3 summarizes some of the measurements made, representing the average, for n from 10 to 16 bits, of the root mean square error to the floating point implementation, $\varepsilon_{n\ bit}$, defined in (3), for a Ω_0 value of 0.05.

Table 3. Average root mean square deviation, in dB, from the fixed to the floating-point implementation to $\Omega_0 = 0.05$.

Order	$\varepsilon^{av}(Q, \Omega_0=0.05)$ [dB]			
	ε_B^{av}	ε_{C1}^{av}	ε_{C2}^{av}	ε_E^{av}
4	90	303	39	254
6	111	193	101	189
8	2502	2500	1105	1124
10	1829	1782	905	1062

The results obtained for ε_{av} at a fixed Ω_0 of 0.05, thus situated in the most troubling zone, have their minimum in the Chebyshev type II filter, which has minimum deviations in every order. Chebyshev type II deviations to the floating-point implementation are presented in Fig. 7.

4.3. SOS Filters' optimization

Table 4 summarizes the coefficients' word length, when optimizing this quantity, for each (Q,Ω_0), to ensure the minimum deviation from the floating-point implementation. It is displayed the average of the coefficients' word length.

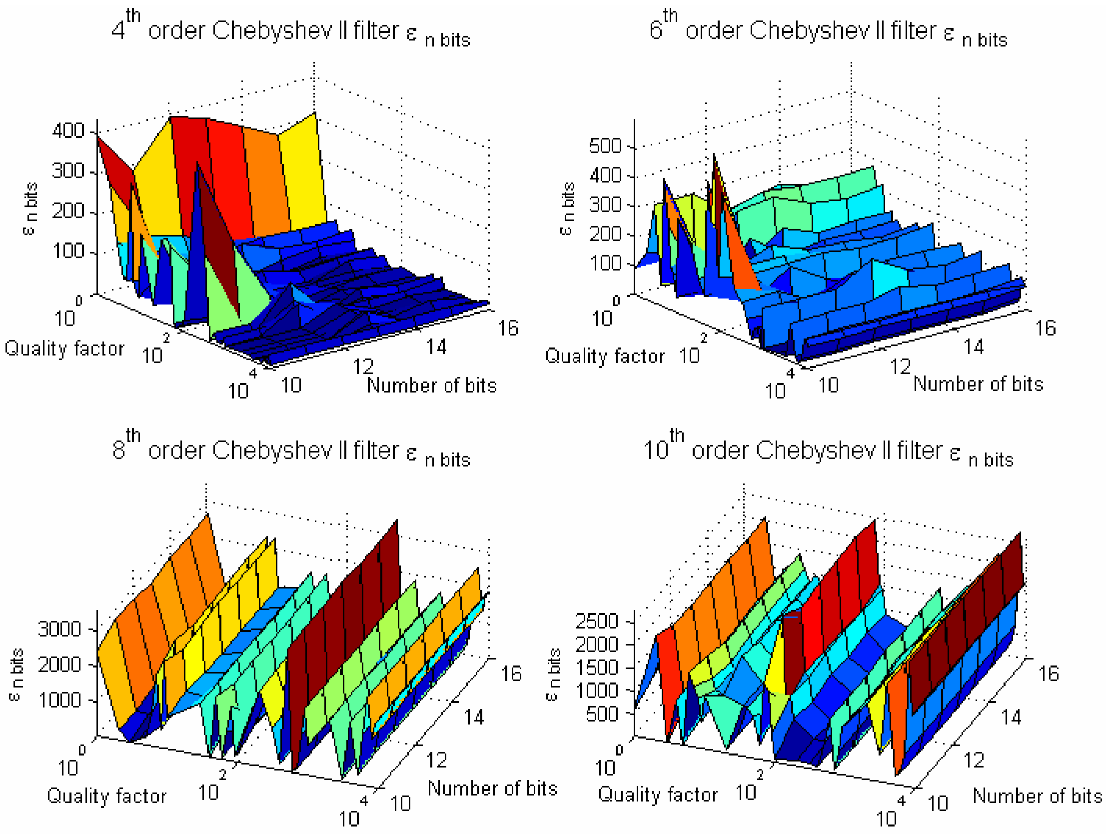

Fig. 7. Higher order Butterworth SOS filters $\varepsilon_{n\ bit}$, with Ω_0 of 0.05, n from 10 to 16 bits, and 37 points Q vector.

Table 4. Number of bits to minimize deviations for all (Q,Ω_0) pairs.

Order	**Average for all (Q,Ω_0)**			
	B	**C1**	**C2**	**E**
4	12.5315	12.7315	12.8568	12.7937
6	11.8360	12.0459	11.8613	12.1514
8	12.0304	12.0802	11.9108	12.0198
10	12.1802	12.2369	12.1477	12.3468

Contrary to what one might *a priori* expect, it is seen that the 4^{th} order has the higher average.

The most demanding filter to minimize the error for all (Q,Ω_0) pairs is the 4^{th} order Chebyshev type II filter. The coefficients' word length dependence on Q and Ω_0 in these cases is shown in Fig. 8.

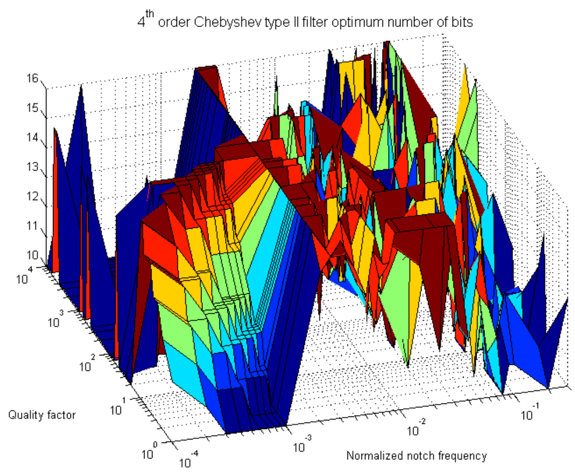

Fig. 8. Optimal coefficients' word length to implement a stable second-order Chebyshev type I filter for the defined Q and Ω_0 vectors minimizing the norm (3).

5. CONCLUSIONS

In the work now reported it was investigated the effect of the design specifications, namely quality factor and normalized cut-off frequency, in the number of significant bits necessary to represent the coefficients of a digital IIR notch filter, and also the deformations the filter suffers with these specifications, when implementing it using fixed-point arithmetic, which has much higher accuracy constrains than the common floating-point implementation.

One important result of our work is that it is forbidden to increase the filter's order above the 2^{nd} if the filter is implemented in a single section. However, the order increase is practically harmless if the filter is decomposed in second-order sections. The simulation results obtained provide comprehensive understanding of the stability requirements and show a critical area of Q and Ω_0 values in which filter's stability is compromised, even for 2nd order. This critical area is especially problematic for biomedical signal processing, since the problematic values of Ω_0 are typical of these applications.

The filters' deviations in the critical zone were measured, and were found to increase significantly when rising above 6th order.

From the classical families of IIR filters, it was seen that Chebyshev type II is the filter family that suffers less with fixed point implementation, and it is also the less demanding in number of significant bits necessary to represent the filters' coefficients.

Digital notch filter behaviour under fixed-point implementation was extensively compared with floating-point under simulation environment. Practical realizations, however, may bring important contributions to the study now reported. Thus, future work must be done regarding the implementation of such filters using namely Digital Signal Processors (DSPs) and Field Programmable Gate Array (FPGAs) evaluating the filters' performances and their deviations and discrepancies from the above presented simulation results.

ACKNOWLEDGMENTS

Eduardo Pinheiro is sponsored by Ph.D. scholarship SFRH/BD/46772/2008 of *Fundação para a Ciência e Tecnologia*.

REFERENCES

[1] H. Cheng and G. Chiu, "Finite Precision Controller Implementation – Limitation on Sample Rate", IEEE/ASME International Conference on AIM 2003, pp. 634-639, Kobe, Japan, 2003.

[2] V. Davídek, M Antosová and B. Psenicka, "Finite Word-Length Effects in Digital State-Space Filters", *Radioengineering*, vol. 8, nº. 4, pp. 7-10, December 1999.

[3] B. Liu, "Effect of Finite Word Length on the Accuracy of Digital Filters – a Review", *IEEE Transactions on Circuit Theory*, vol. CT-18, nº. 6, pp. 670-677, November 1971.

[4] H. Butterweck, J. Ritzerfeld and M. Werter, "Finite Wordlength Effects in Digital Filters: A Review", Fac. Of Elec. Eng., Eindhoven University of Technology, Netherlands, EUT Report 88-E-205, 1988.

[5] T. Laakso, J. Ranta and S. Ovaska, "Design and Implementation of Efficient IIR Notch Filters with Quantization Error Feedback", *IEEE Transactions on Instrumentation and Measurement*, vol. 43, nº. 3, pp. 449-456, June 1994.

[6] R. Otnes and L. McNamee, "Instability Thresholds in Digital Filters Due to Coefficient Rounding", *IEEE Transactions on Audio and Electroacoustics*, vol. AU-18, nº. 4, pp. 456-463, December 1970.

[7] J. Datorro, "The Implementation of Recursive Digital Filters for High-Fidelity Audio", *Journal Audio Engineering Society*, vol. 36, pp. 851-878, November 1988.

[8] M. Er, "Designing Notch Filters with Controlled Null Width", *Signal Processing*, vol. 24, pp. 319-329, September 1991.

[9] T. Laakso and I. Hartimo, "Noise Reduction in Recursive Digital Filters Using Higher-order Error Feedback", *IEEE Transactions on Signal Processing*, vol. 40, pp. 1096-1107, May 1992.

[10] J. Wilkinson, *Rounding Errors in Algebraic Processes*, Englewood Cliffs, New Jersey, 1963.

[11] G. Forsythe and C. Moller, *Computer Solution of Linear Algebraic Systems*, Englewood Cliffs, New Jersey, 1967.

[12] B. Widrow, "Statistical analysis of amplitude quantized sampled-data systems", A*IEE Transactions Appl. Ind.*, vol. 79, pp. 555-568, January 1961.

[13] J. Knatzenelson, "On errors introduced by combined sampling and quantization", IR*E Transactions on Automatic Control*, vol. AC-7, pp. 58-68, April 1962.

[14] J. Knowles and R. Edwards, "Effects of a finite-word-length computer in a sampled-data feedback system", IEE Proceedings., vol. 112, pp. 1197-1207, June 1965.

[15] C. Weaver, J. Van der Groeben, P. Mantey, J. Toole, C. Cole Jr., J. Fitzgerald and R. Lawrence, "Digital Filtering with Applications to Electrocardiogram Processing", *IEEE Transactions on Audio and Acoustics*, vol. AU-16, nº. 3, pp.350-391, September 1968.

[16] K. Liu, R. Skelton and K. Grigoriadis, "Optimal Controllers for Finite Wordlength Implementation", *IEEE Transactions on Automatic Control*, vol. 37, nº. 9, pp. 1294-1304, September 1992.

[17] G. Yan, "New Digital Notch Filter Structures with Low Coefficient Sensitivity", *IEEE Transactions on Circuits and Systems*, vol. CAS-31, nº. 9, pp. 825-828, September 1984.

[18] R. Goodal, "A Practical Method for Determining Coefficient Word Length in Digital Filters", *IEEE Transactions on Signal Processing*, vol. 40, nº. 4, pp. 981-985, April 1992.

[19] J. F. Kaiser, "Digital filters", in *System Analysis by Digital Computers*, F. F. Kuo and J. F. Kaiser, Eds. New York: Wiley, 1966.

[20] P. Mantey, "Eigenvalue sensitivity and state-variable selection", *IEEE Transactions on Automatic Control*, vol. AC-13, nº. 3, pp. 263-269, June 1968.

[21] IEEE, *IEEE 754-2008 Standard for Floating-Point Arithmetic*, August 2008.