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ANALYSIS OF TYPE A UNCERTAINTIES IN PRIMARY ACCELEROMETER CALIBRATIONS APPLYING THE SINE-APPROXIMATION METHOD

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Abstract – This paper discusses the effect of residual components on type A uncertainties determined by multiple linear regression of quadrature homodyne output signals. Primary accelerometer calibrations employing the sine-approximation method allow the determination of the fit uncertainty through the covariance matrix. Frequency components that are not included in the regression model lead to increased uncertainties. The uncertainty obtained from the regression is therefore not exclusively due to random data variance at the calibration frequency, but due to deterministic error of the model. Analysis of the regression residues and a correction method is proposed to obtain more realistic uncertainty values. Some results obtained from simulated data sets are presented.

Keywords: Vibration, Uncertainty, Sine-approximation.

1. BASIC INFORMATION

The sine-approximation method [1] is being widely applied by National Metrology Institutes (NMI) to perform primary calibrations of accelerometers. It is applicable to a wide frequency range and allows determination of the magnitude and phase shift of the complex sensitivity of vibration transducers. It is based on the measurement of the quadrature output signals I and Q of a homodyne interferometer

$$\begin{aligned} I(t) &= \hat{I}\cos\varphi_{Mod}(t) \\ Q(t) &= \hat{Q}\sin\varphi_{Mod}(t) \end{aligned}, \tag{1}$$

where $\hat{I}$ and $\hat{Q}$ denotes the magnitude of the in-phase and the in-quadrature components of the total interferometric phase φ_{Mod}.

An arctangent demodulation scheme can be applied to these signals to obtain the total interferometric phase

$$\varphi_{Mod}(t) = \tan^{-1}\left(\frac{Q(t)}{I(t)}\right), \tag{2}$$

which can by expressed by the discrete relation

$$\varphi_{Mod}(t_i) = [\varphi_0 + \hat{\varphi}_M \cos(\omega_1 t_i + \varphi_s)], \tag{3}$$

where φ_0 is the initial interferometric phase, $\hat{\varphi}_M$ is the magnitude of the modulated phase, ω_1 is the angular frequency of the vibration and φ_s is the displacement phase. This equation can be rewritten as

$$\varphi_{Mod}(t_i) = b_0 + b_1\cos\omega_1 t_i - b_2\sin\omega_1 t_i \tag{4}$$

and the unknown coefficients b can be obtained by a multiple linear regression. The sine-approximation method refers to fitting a sine wave to the experimental data of the total interferometric phase.

Equation (4) can be written in matrix format

$$\mathbf{Y} = \mathbf{X}\beta + \varepsilon, \tag{5}$$

where $\mathbf{Y}$ is a $n\times 1$ vector containing n observations of the dependent variable $\varphi_{Mod}(t_i)$. $\mathbf{X}$ is a matrix of dimension $n\times 3$ with known shape, β is a 3×1 vector of independent variables b and ε is a $n\times 1$ vector of errors.

$$\begin{bmatrix} \varphi_{Mod}(t_0) \\ \varphi_{Mod}(t_1) \\ \varphi_{Mod}(t_2) \\ \vdots \\ \varphi_{Mod}(t_N) \end{bmatrix} = \begin{bmatrix} 1 & \cos\omega_1 t_0 & -sen\omega_1 t_0 \\ 1 & \cos\omega_1 t_1 & -sen\omega_1 t_1 \\ 1 & \cos\omega_1 t_2 & -sen\omega_1 t_2 \\ \vdots & \vdots & \vdots \\ 1 & \cos\omega_1 t_N & -sen\omega_1 t_N \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} + \begin{bmatrix} \varepsilon_0 \\ \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_N \end{bmatrix} \tag{6}$$

Both β and ε are unknown in this equation, but it is β for which we want to solve the problem. The solution by the least squares method [2] [3] furnishes the vector **b** as the best estimate of β. The vector **b** is calculated by

$$\mathbf{b} = (\mathbf{X^T X})^{-1}\mathbf{X^T Y}. \tag{7}$$

The amplitude of the modulated phase $\hat{\varphi}_M$ and the displacement phase φ_s can be calculated by the equations

$$\hat{\varphi}_M = \sqrt{b_1^2 + b_2^2}, \tag{8}$$

$$\varphi_s = \tan^{-1}\left(\frac{b_2}{b_1}\right). \tag{9}$$

The amplitude of the displacement is then determined by

$$\hat{s} = \frac{\lambda}{4\pi}\hat{\varphi}_M \,. \tag{10}$$

According to eq. (8), the combined standard uncertainty of the magnitude of the modulated phase is

$$u_c(\hat{\varphi}_M) = \frac{1}{\hat{\varphi}_M}\left[b_1^2 u^2(b_1) + b_2^2 u^2(b_2) + 2 b_1 b_2 \operatorname{cov}(b_1, b_2)\right]^{1/2}, \tag{11}$$

where $u^2(b_1)$ and $u^2(b_2)$ are the variances and $\operatorname{cov}(b_1, b_2)$ is the covariance associated to the coefficients b_1 and b_2. These values are given by the variance-covariance matrix **V(b)**

$$\mathbf{V(b)} = \left(\mathbf{X^T X}\right)^{-1}\sigma^2 = \begin{bmatrix} V_{00} & V_{01} & V_{02} \\ V_{10} & V_{11} & V_{12} \\ V_{20} & V_{21} & V_{22} \end{bmatrix}, \tag{12}$$

where σ^2 is an estimation of the quadratic arithmetic mean of the residues of the regression, obtained by dividing the quadratic mean of the residues by (n-3) degrees of freedom.

$$\sigma^2 = \frac{(\mathbf{Y^T Y} - \mathbf{b^T X^T Y})}{n-3} \tag{13}$$

Then, the variances of coefficients b are obtained from the diagonal terms of matrix **V(b)**, where $u^2(b_0) = V_{00}$, $u^2(b_1) = V_{11}$ and $u^2(b_2) = V_{22}$ and the covariances are obtained from the off-diagonal terms $\operatorname{cov}(b_1, b_2) = V_{12} = V_{21}$.

The expanded uncertainty of $\hat{\varphi}_M$ is calculated by multiplying the combined standard uncertainty $u_c(\hat{\varphi}_M)$ by a coverage factor k_p chosen on the basis of a level of confidence required for the interval.

$$U(\hat{\varphi}_M) = k_p\, u_c(\hat{\varphi}_M) \tag{14}$$

The determination of the uncertainty through the analysis of the residues requires a careful analysis to assure that the regression model is representative of the data. This condition requires that the residues have a null mean and a variance σ^2.

It should be highlighted that even if we have a perfect identification of the target amplitude and phase, the uncertainty given directly by regression may not represent this match. This may happen because the uncertainty depends on how the fit model represents not only the waveform at the frequency of interest (i.e. the driving frequency in the case of the SAM) but the observed signal as a whole. The uncertainty may include, besides the component due to the random variance of the observed data, an undesired systematic error of the model (bias).

If there are other periodic components significant compared to the amplitude of the motion at excitation frequency f_1, and are not included in the regression model, it will occur an over estimation of the uncertainty, because the variance **V(b)** depends directly on the variance of the residues σ^2. This effect typically occurs due to hum, non-linearities as harmonic distortion, structural vibrations, inadequate vibration isolation, etc.

Dobodsz *et al.* [4] have suggested to consider a normal distribution of the residues with a standard distribution of $b/\sqrt{3}$, where $\pm b$ represents the limits within the residues are included. This method can be applied to evaluate the uncertainty of the amplitude of the total interferometric phase $\hat{\varphi}_M$, but not for the displacement phase φ_s. Even in the case of amplitude, it is necessary to establish the limits $\pm b$ through evaluation of the random dispersion around the remaining deterministic signals present in the residues.

An alternative method has been evaluated through computational simulations with very good results. It consists in adapting the regression model in order to include all the main spectral components present in the signal. In the specific case of the sine-approximation method, this alternative may not be feasible to be applied due to the large amount of data involved. It is usual to have vectors of observations in the order of 10^5 to 10^6 being processed. The increase of the dimensions of matrix **X** requires an increase in computational power, which may turn the processing of the regression unviable. Each new spectral component added to the model would require the addition of two new columns to matrix **X.**

A more versatile method is proposed in the next section.

2. PROPOSED METHOD

The approach proposed in this paper consists in correcting the deterministic error from the uncertainty determined by linear regression.

First, a calibration must be run with the determination of the uncertainties for amplitude and phase shift estimated by multiple linear regression. Then the residues shall be analyzed. This can be done by a set of statistical tools, which were implemented in LabVIEW environment. The program developed provides the following graphs for analysis:

- Residues x time,
- Residues x estimated regression value
- Histogram of residues normalized by the standard deviation
- Residues normalized by the standard deviation x cumulative percentage
- Residues x frequency

These graphs allow the user to visually check the distribution of the residues and to verify how close they are to a normal distribution N(0, σ^2). They provide evidences that help the evaluation of the degree of inadequacy of the SAM model and of the overestimation of uncertainty.

The residues can also be analyzed in the frequency domain. This allows the non-modeled spectral components, which are significant in the residues composition to be detected easily. After applying the Fourier transform to the residues, a peak-picking algorithm is applied for identification of the amplitudes of major spectrum peaks and their corresponding frequency values.

After the identification of the main secondary spectral components responsible for the distortion of the residue, it is possible to generate a deterministic multi-tone signal with

null variance, combining these secondary components with the fundamental component. A new regression using the same model as used before can then be applied to this simulated signal. The magnitude and phase uncertainty values obtained for the simulated signal can then be used for correction of the uncertainty obtained for the observed data by the relation

$$u(\Psi_{cor}) = \sqrt{u^2(\Psi_{exp}) - u^2(\Psi_{sim})}, \qquad (15)$$

where

$u(\Psi_{exp})$ – amplitude or phase uncertainty estimated by the SAM applied to the observed data,

$u(\Psi_{sim})$ – amplitude or phase uncertainty estimated by the SAM applied to the multi-tone simulated data,

$u(\Psi_{cor})$ – corrected amplitude or phase uncertainty.

It should be noted that the multi-tone simulated signal vector generated for the correction applying eq. (15) requires the use of the same sampling rate and number of samples as in the observed data vector. This procedure is important to maintain the same conditions during both regressions.

3. EXAMPLES

Two examples are presented in this section to show how the computational program developed at Inmetro is used to perform the statistical analysis of the sine-approximation method. The correction method proposed in this paper is applied to the second example, in order to demonstrate its feasibility.

The signals used in the two following examples were generated using the following equation

$$A(t) = \sum_{i=1}^{m} \hat{A}_i \sin(2\pi f_i t + \frac{\pi}{90}\theta_i) + \hat{w} \cdot n(t). \qquad (16)$$

This equation shows that the signals simulated were composed of a sum of sine waves and noise. In our case, the noise term was assumed to be a uniformly distributed pseudorandom noise with amplitude $\hat{w}$. This noise was obtained by multiplying the output of a noise generator $n(t)$, which varies in the range [-1, 1] by a scaling constant $\hat{w}$.

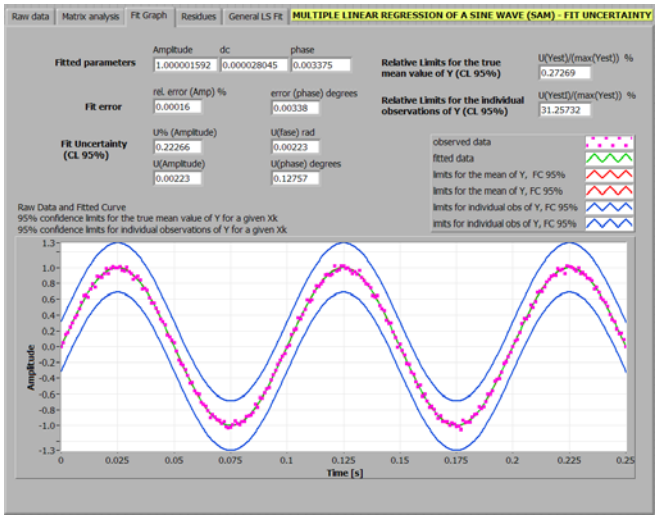
3.1. Example – Sine wave plus noise (WF1)

The aim of this first example is to present the output of the computational program when applied to a single sine wave embedded in pure random noise. The waveform WF1 used for this purpose is composed of a single unit-amplitude 10 Hz sine wave and 0.05 amplitude noise.

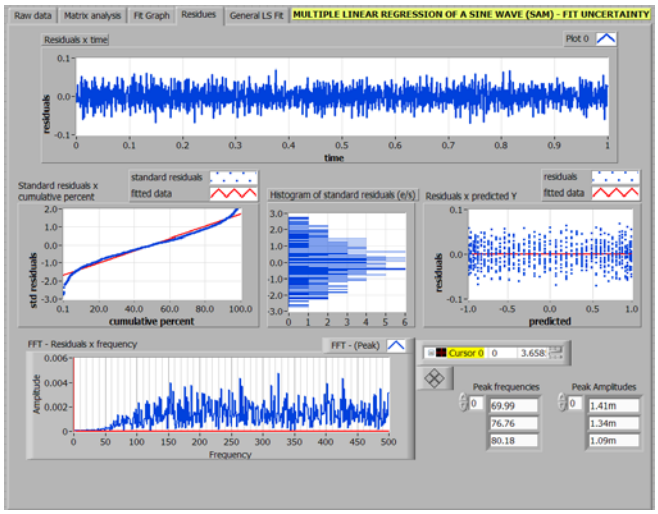
Two panels of the computer program are presented in Fig.1.

Fig. 1 (a) shows the panel "Fit Graph", which includes the regression parameters magnitude, phase and dc offset obtained by applying the sine-approximation method to the observed signal, the amplitude and phase uncertainties estimated and a graph including the observed data, the fitted sine wave, the limits of the true mean and the limits for individual observations of a given input assuming a level of confidence of 95%.

In this example, a 0.00006% relative error was verified in the fitted amplitude value and a 0.23% relative uncertainty was obtained for a 95% confidence level.



(a) Screen "Fit Graph"



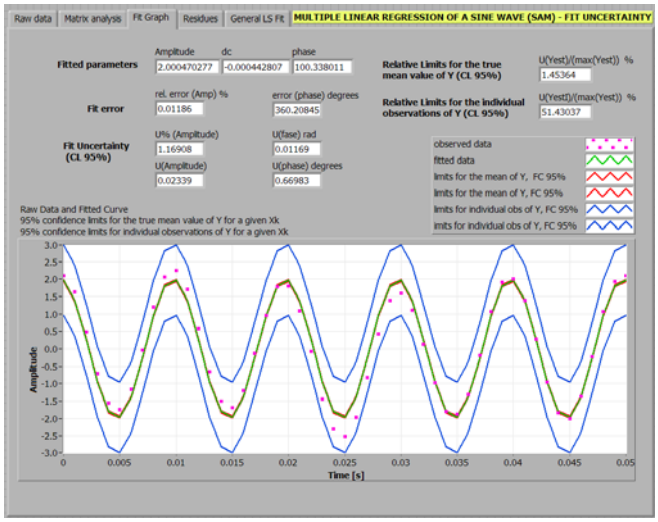
(b) Panel "Residues"

Fig. 1. Program for analysis of the SAM uncertainty - Simulated input signal: $\hat{A}_1 = 1$, f_1 = 10 Hz, noise amplitude $\hat{w}$ = 0.05; sampling configuration: Fs=1000, N=1000.

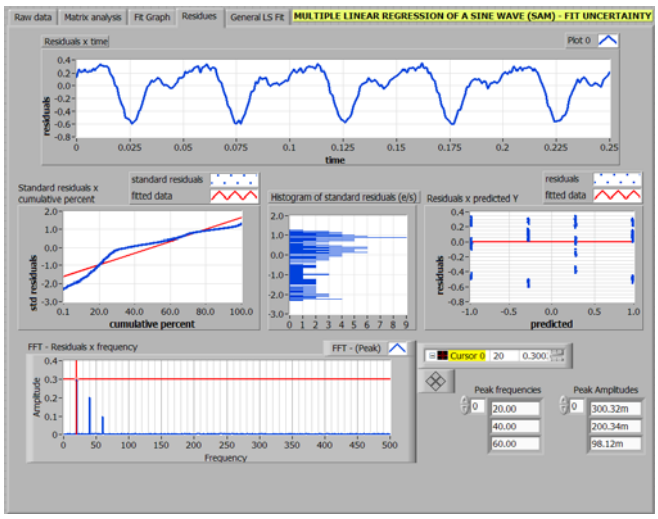
The panel "Residues" presented in Fig. 1 (b) includes diverse graphical representations of the regression residues. The graph Residuals x Time demonstrate that the distribution of the residues against time for this first example show an adequate aspect of randomness. This is confirmed by a nearly Gaussian symmetric distribution within the limits of $\pm 3\sigma$ in the Histogram of standard residuals. In addition, the plot Standard residuals x Cumulative percent show that the standard residuals are close to the straight line of the fitted data and the plot Residuals x predicted Y gives evidence of random distribution of the points. In the frequency domain there is no evidence of any considerable spectral components in the residues.

3.2. Example II – Multi-sine wave plus noise (WF2)

In order to exemplify the effect caused by secondary spectral components on the sine fit uncertainty, a waveform WF2 composed of a primary sine wave plus three additional secondary sinusoidal components and random noise will be used. The signal was generated using eq. (15) with the following parameters: $\hat{A}_1 = 2$, $\hat{A}_2 = 0.3$, $\hat{A}_3 = 0.2$, $\hat{A}_4 = 0.1$, $f_1 = 100$ Hz, $f_2 = 20$ Hz, $f_3 = 40$ Hz, $f_4 = 60$ Hz, $\theta_1 = 100°$, $\hat{w} = 0.05$. A sampling rate of 1000 samples/s and 1000 samples were used to generate the signal. The results obtained for this signal WF2 are presented in Fig. 2.



(a) Panel "Fit Graph"



(b) Panel "Residues"

Fig. 2. Program for analysis of the SAM uncertainty - Simulated input signal WF2: $\hat{A}_1 = 2$, $\hat{A}_2 = 0.3$, $\hat{A}_3 = 0.2$, $\hat{A}_4 = 0.1$, $\theta_1 = 100°$, noise amplitude $\hat{w} = 0.05$; $f_1 = 100$ Hz, $f_2 = 20$ Hz, $f_3 = 40$ Hz, $f_4 = 60$ Hz; sampling configuration: Fs=1000, N=1000.

Fig. 2 (a) shows that the amplitude value determined by applying the multilinear regression to signal WF2 was 2.0047. This result represents a 0.024% error relative to the amplitude originally used to generate the signal (i.e. $\hat{A}_1 = 2$). On the other hand, a quite high uncertainty of 1.17% is obtained for a 95% confidence level. This increased uncertainty is caused by the lack of agreement between the single-sine regression model and the actually observed signal.

In Fig. 2(b), the lack of fit is evident in the different graphs. The presence of spectral components in the residue is visible in the frequency domain, with the respective frequencies and amplitudes being presented in the tables in the lower right part of the panel. The periodic behavior of the residuals is also easily detected in the plot Residuals x time.

The amplitude and phase results and the associated uncertainties determined for this example are presented in the Table 1. The results obtained for the original observed signal WF2, are presented in the first row and for the multi-tone simulated signal (MTS), generated according to the correction procedure proposed are presented in the second row. The multi-tone signal MTS was generated using four sine waves with amplitude and frequency values corresponding to the primary component estimated by the sine-approximation method (shown in Fig. 2(a)) and to the three remaining secondary components observed in the residues. These values were taken from the two tables in the lower right part of the panel Residues, shown in Fig.2 (b). The parameters of the multi-tone simulated MTS signal generated are compiled in Table 2.

Table 1. Results of the procedure applied to a multi-tone waveform plus noise

Signal	$\hat{A}_1$	$U(\hat{A}_1)/\hat{A}_1$ [%]	θ_1 [°]	$U(\theta_1)$ [°]
WF2	2.00047	1.16908	100.34	0.670
MTS	2.00047	1.16213	100.34	0.666
Corrected uncertainty		0.12729		0.0728
REF	1.99922	0.12705	99.93	0.0729

Table 2. Parameters of the multi-tone simulated signal generated for correction of the uncertainty values.

Composition of MTS	Frequency [Hz]	Amplitude	Phase [°]
$\hat{A}_{sim1}(t)$	100	2.00047	100.34
$\hat{A}_{sim2}(t)$	20	0.30032	0
$\hat{A}_{sim3}(t)$	40	0.20034	0
$\hat{A}_{sim4}(t)$	60	0.09812	0

The results in Table 1 show that the relative uncertainty of amplitude was reduced from 1.17% to 0.13% and the uncertainty of phase was reduced from 0.67° to 0.07° after applying the correction procedure proposed in this paper. Therefore, a reduction of one order of magnitude was obtained for both amplitude and phase uncertainties in this example. These corrected uncertainties are very close to the

ones obtained when applying the sine approximation method to the waveform REF, generated exclusively with the primary component and noise. The parameters used to generate REF were the same ones used for the primary component in WF2: $\hat{A}_1 = 2$, $\theta_1 = 100°$, $\hat{w} = 0.05$. Table 1 includes the results obtained for REF in row 5.

4. CONCLUSIONS

The analysis of the residues is currently not properly considered by many users of the sine-approximation method. Common statistical tools and visual analysis of graphs in time and frequency domain may provide valuable information about the calibration system and about the quality of the results obtained.

The sine-approximation method is based on the fit of a sine wave to experimental data. The basic assumption is that the observed data from the homodyne interferometer and from the accelerometer under test can be represented by a single sine wave with frequency equal to the oscillatory motion plus some random noise. Unfortunately, secondary spectral components may be embedded in the observed data due to the presence of harmonic distortion, hum, relative motion between seismic blocks, shaker resonances, etc.

The lack of agreement between the regression model and the observed data reflects on increased uncertainties obtained from the variance matrix.

The method proposed allows the correction for obtaining a more realistic evaluation of the type A uncertainty determined by the sine-approximation method.

The correction can be applied to the uncertainty of both magnitude and phase of the fitted sine wave. The same methodology described herein can be extended to many other analysis of linear regressions, where the model can not be properly adapted to better fit the observed data.

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