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VALIDITY OF POLYNOMIALS AS RESULTS FOR COMPARISONS

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Abstract – Guidelines provided by [1] have been used worldwide to evaluate key comparisons. The aim of this paper is to demonstrate the validity of these procedures when polynomials instead of fixed values are provided as comparison results.

Keywords: comparison, polynomial.

1. INTRODUCTION

Luckily most transfer standards materialise one measurement value each, but there many cases where the travelling standard is a measurement instrument that materialises a range of values. Typical examples are a piston-cylinder assembly for pressure measurements or a weighing instrument, but there many of them through all the branches of metrology. Up to now comparisons have been performed for fixed values of the standard whole range, but what is really useful and characterises the standard is a function, which is a polynomial most times. In fact, a polynomial is quite a general function because any function can be “expressed” as a polynomial by means of a Taylor series expansion. In practise, most instruments are developed in order to be linear, so that the polynomial case is more general than the one that is usually found, the straight line.

2. DESCRIPTION

Guidelines provided by [1] describe two procedures. This paper is only going to take into account procedure A which is the most common one and is the recommended one by the guidelines themselves.

According to the Mutual Recognition Arrangement (MRA) [2] in order to evaluate a comparison the following parameters have to be determined:

1. The degree of equivalence of each laboratory, which is expressed quantitatively by two terms: its deviation from the key comparison reference value and the uncertainty of this deviation at the 95 % level of confidence.

2. The degree of equivalence between laboratories is expressed quantitatively by two terms: the difference of their deviations from the comparison reference value and the uncertainty of this difference at the 95 % level of confidence.

2.1. Method

Each laboratory i, ($i = 1,\dots, N$) will provide a polynomial $p_i(x)$ and its associated uncertainty $u_i(x)$ where x is the input quantity.

$$p_i(x)=a_{i0}+a_{i1}\cdot x+a_{i2}\cdot x^2+\dots+a_{ik}\cdot x^k \quad (1)$$

$$u_i(x) = b_{i0}+b_{i1}\cdot x+b_{i2}\cdot x^2+\dots+b_{ih}\cdot x^h \quad (2)$$

The following steps have to be performed in order to evaluate the comparison:

1. Determine the weighted mean of the laboratories’ measurements, using the inverses of the squares of the associated standard uncertainties as the weights:

$$y(x)=\frac{p_1(x)/u_1^2(x)+\dots+p_N(x)/u_N^2(x)}{1/u_1^2(x)+\dots+1/u_N^2(x)} \quad (3)$$

2. Determine the standard deviation $u_y(x)$ associated with $y(x)$ from (4).

$$\frac{1}{u_y^2(x)}=\frac{1}{u_1^2(x)}+\dots+\frac{1}{u_N^2(x)} \quad (4)$$

3. Apply a chi-squared test to carry out an overall consistency check of the results obtained:

- Form the observed chi-squared value.

$$\chi^2_{obs}=\frac{(p_1(x)-y(x))^2}{u_1^2(x)}+\dots+\frac{(p_N(x)-y(x))^2}{u_N^2(x)} \quad (5)$$

- Assign the degrees of freedom (N is the number of laboratories).

$$\nu = N-1 \quad (6)$$

- Regard the consistency check as failing if (7).

$$Pr\{\chi^2(\nu)>\chi^2_{obs}\}<0{,}05 \quad (7)$$

4. If the consistency check fails functions that fulfil condition (8) will be classified as discrepant at 5% level of significance.

$$|p_i(x)-y(x)|/U_{di}(x)>1 \quad (8)$$

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If the consistency check does not fail, $y(x)$ will be accepted as the reference value and $u_y(x)$ will be accepted as its standard uncertainty. Besides, the degrees of equivalence can be calculated.

The degree of equivalence of laboratory i will be the pair of functions $(d_i(x), U_i(x))$ given by(9) and (10).

$$d_i(x) = p_i(x) - y(x) \quad (9)$$

$$U_i(x) = 2\sqrt{u_i^2(x) - u_y^2(x)} \quad (10)$$

The previous formula involves a *difference* of two variances as a consequence of the mutual dependence of $u_i^2(x)$ and $u_y^2(x)$.

The degree of equivalence between laboratory i and laboratory j will be given by the pair of values $(d_{i,j}(x), U_{i,j}(x))$ using (11) and (12).

$$d_{i,j}(x) = p_i(x) - p_j(x) \quad (11)$$

$$U_i(x) = 2\sqrt{u_i^2(x) + u_y^2(x)} \quad (12)$$

It is obvious that all the previous treatment depends on x. This means that one laboratory can be discrepant for some range of x but it is not for the whole range. That is the power of the method.

2.2. Validity

In order to use this method the following conditions have to be applied according to [1]:

1. The travelling standard has good short-term stability and stability during transport.

2. Each participating laboratory's measurement is realized independently of the other laboratories' measurements in the comparison.

3. For each laboratory a Gaussian distribution (with mean equal to the laboratory's measurement and standard deviation equal to the provided associated standard uncertainty) can be assigned to the measurand of which the laboratory's measurement is an estimate.

If polynomial functions are used instead of functions there is no change for conditions 1 and 2. Condition 3 is the one that has to be checked.

The first question must be how these polynomials have been obtained. It is clear that an approximation technique based on the "minimum χ^2" approach has to be used. This is a very general method; a possible reference is [3]. In this approximation all calculations to obtain the polynomial are linear and, as a general property of Gaussian distributions is that any linear combination of Gaussian distributions is also a Gaussian distribution [4], it is clear that condition 3 is fulfilled and procedure A in [1] can be used for polynomials.

3. EXAMPLE

In order to evaluate the validity of this procedure as well as its advantages a comparison in the pressure field will be evaluated. Its transfer standard is a piston cylinder assembly and the parameter that characterises it is called "effective area" A_p. A piston cylinder assembly is basically a piston that rotates freely inside a cylinder. The pressure is applied in the lower part of the assembly and the piston is loaded with masses. When the piston is under floating equilibrium the downward gravity force, which is caused by the masses on the piston, is equal to the upward force, which is exerted by the pressure on the assembly. Under these equilibrium conditions the pressure can be determined as the ratio between the gravity force and the effective area. The effective area itself depends on pressure. Most times it is enough to consider a linear dependence on pressure like (13), where A_0 is the effective area for zero pressure, P is the nominal pressure and λ is the pressure distortion coefficient.

$$A_p = A_0 \cdot (1 + \lambda \cdot P) \quad (13)$$

Equation (13) is the result provided to customers in a piston-cylinder assembly calibration certificate, but in order to use procedure A in [1] in comparisons each laboratory provides different effective area values for different nominal pressures.

In this example the same comparison is going to be evaluated both ways, with fixed values and polynomials.

Table 1. Table of values provided for the comparison. Every laboratory has provided effective areas A_p with their uncertainties u_p (k = 1) for common nominal pressures.

Nominal pressure (kPa)	Lab 1		Lab 2		Lab 3	
	A_p (mm^2)	u_r (x10^6)	A_p (mm^2)	u_r (x10^6)	A_p (mm^2)	u_r (x10^6)
100	980,5531	7	980,5464	6,9	980,5663	15
200	980,5855	7	980,5701	6,6	980,5808	15
400	980,6604	6,6	980,659	6,4	980,6561	12
600	980,717	6,5	980,744	6	980,7498	10
800	980,798	6,3	980,8146	6	980,8082	10
1000	980,892	6,3	980,9038	6	980,8994	10

Nominal pressure (kPa)	Lab 4		Lab 5	
	A_p (mm^2)	u_r (x10^6)	A_p (mm^2)	u_r (x10^6)
100	980,5525	7,2	980,5423	11
200	980,5976	6,9	980,576	10
400	980,648	6,8	980,6491	9,6
600	980,7427	6,8	980,7351	9,6
800	980,8111	6,7	980,7983	9,3
1000	980,89	6,6	980,8917	9,3

Nominal pressure (kPa)	Lab 6		Lab 7	
	A_p (mm^2)	u_r (x10^6)	A_p (mm^2)	u_r (x10^6)
100	980,559	8,1	980,5546	13
200	980,5985	8,1	980,5922	12
400	980,6585	7,9	980,6524	10
600	980,7388	7,8	980,7326	10
800	980,809	7,8	980,8029	10
1000	980,8993	7,8	980,8731	10

If values in table 1 are evaluated according to procedure A in [1] these are the results which are obtained (steps 1 and 2):

Table 2. Table of results obtained in steps 1 and 2 according to procedure A in [1].

Nominal pressure (kPa)	Weighted mean of A_p (mm^2)	Standard deviation of the weighted mean of u_p (mm^2)
100	980,5522	0,0031
200	980,5858	0,0030
400	980,6573	0,0029
600	980,7365	0,0028
800	980,8068	0,0028
1000	980,8900	0,0028

The observed chi-squared values are in Table 3 (the degrees of freedom are 6):

Table 3. Results of the evaluation according to step 3.in procedure A en [1].

Nominal pressure (kPa)	Observed chi-squared value	$Pr\left\{\chi^2(\nu) > \chi^2_{obs}\right\}$
100	3,3	0,772
200	12,9	0,045
400	1,4	0,968
600	14	0,030
800	5,3	0,501
1000	7,5	0,278

As it can be seen in table 3 for 200 kPa and 600 kPa the probability is less than 0,05, so the consistency check fails.

Table 4 shows the evaluation of condition (8) for each laboratory:

Table 4. This table shows the evaluation of condition (8).

Nominal pressure (kPa)	Lab 1	Lab 2	Lab 3	Lab 4	Lab 5	Lab 6	Lab 7
100	0,07	0,49	0,49	0,02	0,48	0,47	0,10
200	0,02	1,38	-0,17	0,98	0,53	0,87	0,28
400	0,27	0,15	-0,05	0,06	0,46	0,08	0,26
600	1,70	0,73	0,71	0,51	0,08	0,16	0,21
800	0,80	0,75	0,07	0,36	0,49	0,15	0,21
1000	0,73	0,37	0,50	0,00	0,10	0,65	0,90

As it can be seen Laboratory 1 is discrepant for 600 kPa and Laboratory 2 is discrepant for 200 kPa. From the metrological point of view these results do not say much. In fact there is no difference in the procedure, the operator, etc; so that it is difficult to find the reasons for these results if the measurement conditions where the same.

The second part of this example is the comparison evaluation applied to the linear regressions of effective area versus pressure (table 5).

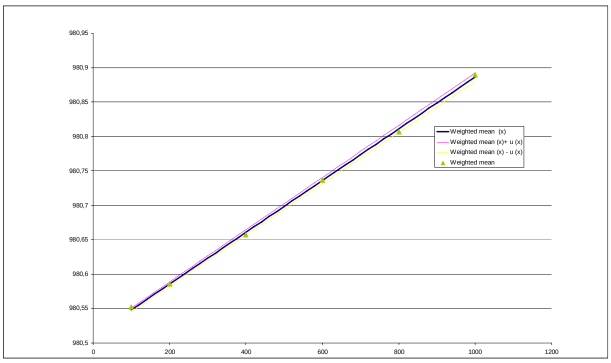
Table 5. Table of the linear regressions of the effective area versus pressure with their uncertainty equations (k = 1). These equations have been obtained with the data in table 1.

	Effective area equation (mm^2, P in kPa)	Effective area uncertainty equation (mm^2, P in kPa)
Lab 1	A(P)= 980,5126·(1+ 3,7x10^{-7}·P)	u^2(P)=2,7x10^{-5}+7,5x10^{-8}·P+6,9x10^{-11}·P^2
Lab 2	A(P)= 980,5010·(1+ 4,0x10^{-7}·P)	u^2(P)=2,6x10^{-5}+7,0x10^{-8}·P+6,4x10^{-11}·P^2
Lab 3	A(P)= 980,5137·(1+ 3,9x10^{-7}·P)	u^2(P)=1,1x10^{-4}+2,9x10^{-7}·P+2,4x10^{-10}·P^2
Lab 4	A(P)= 980,5170·(1+ 3,8x10^{-7}·P)	u^2(P)=2,8x10^{-5}+7,9x10^{-8}·P+7,4x10^{-11}·P^2
Lab 5	A(P)= 980,4991·(1+ 3,9x10^{-7}·P)	u^2(P)=6,1x10^{-5}+1,7x10^{-7}·P+1,5x10^{-10}·P^2
Lab 6	A(P)= 980,5182·(1+ 3,8x10^{-7}·P)	u^2(P)=3,8x10^{-5}+1,0x10^{-7}·P+1,0x10^{-10}·P^2
Lab 7	A(P)= 980,5171·(1+ 3,6x10^{-7}·P)	u^2(P)=8,1x10^{-5}+2,2x10^{-7}·P+1,9x10^{-10}·P^2

These equations in table 5 are the ones that laboratories provide in their certificates. As they are provided in their certificates it is clear these equations are the proper measurement results, the ones that should be compared.

In this case applying (3), (4) and (5) not single values but functions are obtained as shown in plot 1.

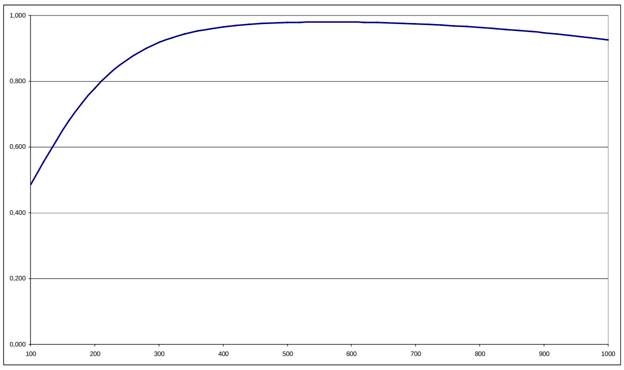
Plot 1. Weighted mean function and this function plus and minus its uncertainty function ((1) and (2)). The green triangles are the weighted means obtained from fixed values.



The weighted mean function (plot 1) looks like a straight line although its functional relation is something much more complicated. In green the weighted means obtained by the previous method are also showed. The uncertainties are small enough not to be seen on the plot.

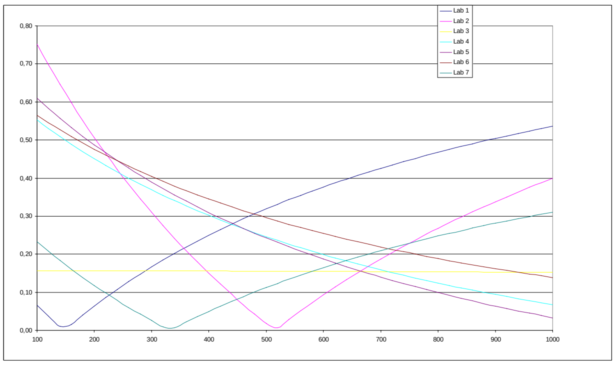
The third step is the chi-squared test evaluation. Its results are showed in plot 2.

Plot 2. It shows the probability that the theoretical chi squared value χ^2 (ν) is more than the calculated chi squared value χ^2_{obs} .



It is clear from plot 2 that condition (7) is never fulfilled, so that all the results are consistent. The evaluation of condition (8) does not provide fixed values any more. Now a function is obtained for every laboratory as shown in plot 3.

Plot 3. Evaluation of condition (8) for every laboratory.



It is clear than all these functions are more than 1 and condition (8) is always fulfilled.

The degrees of equivalence of each laboratory and between laboratories can also be evaluated as functions without loss of generality.

This example has shown that the evaluation of polynomials instead of fixed values is more realistic and provides more coherence to the results. This is obvious because the instrument behaviour is better expressed by means of polynomials instead of different fixed values. On the other hand, there is an agreement between the weighted means obtained by both methods according to plot 1, so the reference values are “basically the same”.

4. CONCLUSIONS

The real characterization of many measurement standards follows a polynomial; that is the reason why there is a lack of information when only some corrections for fixed values are provided. This paper tries to demonstrate that polynomials can be compared in the same way as fixed values using the procedure A given in [1].

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