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APPROXIMATE GCD OF INEXACT UNIVARIATE POLYNOMIALS

Pablo Lecumberri [1], *Marisol Gómez* [1], *Alfonso Carlosena* [2]

[1] Department of Mathematics, Universidad Pública de Navarra, Pamplona, Spain
[2] Department of Electrical and Electronic Engineering, Universidad Pública de Navarra, Pamplona, Spain

Abstract – The problem of finding the greatest common divisor (GCD) of univariate polynomials appears in many engineering fields. Despite its formulation is well-known, it is an ill-posed problem that entails numerous difficulties when the coefficients of the polynomials are not known with total accuracy, as, for example, when they come from measurement data. In this work we propose a novel GCD estimation method designed to cope with such inaccuracies. An example of recovery of transient impulsive signals is provided to show the performance of the proposed method working on measurement data.

Keywords: Greatest common divisor, deconvolution

1. INTRODUCTION

The ever-increasing presence of polynomial models in almost every engineering field has prompted a renewed interest in polynomial methods and operations in recent years. One fundamental operation that has a wide range of applications is the determination of the GCD a set of polynomials. It is a useful tool in the study of linear systems [1], computer vision, image processing [2], computer-aided design [3] and system identification [4], among other fields.

If the polynomials are known exactly and symbolic computations are used, long-known methods, such as Euclid's algorithm, yield the correct GCD of the set. However, data inaccuracy is unavoidable, specially when it comes from measurements. In this situation, the exact GCD (EGCD) of the set is 1. This is due to the ill-posedness of the GCD computation problem. The solution is not continuous on the input data, and even slight perturbations make polynomials with non-constant GCD coprime.

Two definitions of approximate GCD (AGCD) have been proposed to find the GCD of sets of polynomials with inexact coefficients. Both of them follow a common strategy for solving ill-posed problems: The AGCD is defined as the EGCD of another set of polynomials which is closest to the input set in some sense, fulfilling some condition:

- The d-AGCD is found when the degree of the AGCD is restricted to be equal to some integer d.
- For ε-AGCD, the degree of the AGCD is set to the maximum degree of the EGCD of all polynomial sets within a distance ε of the input set.

Most works on ε-AGCD first estimate the degree of the AGCD and then perform a d-AGCD computation. Techniques for AGCD estimation include decomposition of matrices [5,6], optimization [7,8,9,10] and root grouping [11]. Some of the proposed methods work only on sets of two polynomials, and most of them are impractical for computing the ACGD of many polynomials of large degree due to their computational burden.

In this work we fully develop a novel d-AGCD computation method suitable for sets of many polynomials of large degree. Starting from the conditioned optimization problem suggested by the d-AGCD definition, Section 2 shows the transformations that lead to the formulation of the problem as a simple sequential unconstrained quadratic minimization. The proofs for all lemmas can be found in [12]. Section 3 shows an application of AGCD computation to the indirect measurement of impulsive transient signals through a blind deconvolution operation.

The notation used in this work is fairly conventional. Vectors and matrices are represented by bold lower case and uppercase letters respectively. Polynomials and scalars are written in normal font-weight. The complex conjugate of a scalar or matrix is denoted by a * superscript, while T and H denote respectively the transpose and complex conjugate transpose of a matrix. The Moore-Penrose pseudoinverse of a matrix is represented with a + superscript . Superscripts (r) and (i) refer to the real and imaginary part of a scalar or vector.

2. AGCD COMPUTATION METHOD

2.1. Preliminaries

The proposed AGCD method builds on the vectors and matrices introduced by the following definitions.

Definition 2.1 (*Modes*). Consider the column-vector $\mathbf{m}(x) \in \mathcal{M}_{\mathbb{C}}(m+1,1)$ defined as

$$\mathbf{m}(x) = \begin{bmatrix} 1 & x & x^2 & \cdots & x^m \end{bmatrix}^{\mathrm{T}}. \quad (1)$$

A mode of root $z \in \mathbb{C}$ and order 0, $\mathbf{m}_{z,0} \in \mathcal{M}_{\mathbb{C}}(m+1,1)$, is defined as $\mathbf{m}_{z,0} = \mathbf{m}(x)$. A mode of root $z \in \mathbb{C}$ and order $k > 0$ is the vector $\mathbf{m}_{z,k} = \left. \frac{\delta^k \mathbf{m}(x)}{\delta^k x} \right|_{x=z}$.

Definition 2.2 (*Associate modes*). We say a mode $\mathbf{m}_{z,k}$ is an associate mode of a polynomial $a(x)$ if z is a root of $a(x)$ with multiplicity greater than k.

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Definition 2.3 (*Coefficients matrix*). Given a set $\left\{ a_i(x) = a_0^{(i)} + \ldots + a_m^{(i)} \cdot x^m \;\middle|\; \deg(a_i(x)) \le m,\ i = 1,\ldots,Q \right\}$, its coefficients matrix $\mathbf{A} \in \mathcal{M}_{\mathbb{C}}(m+1,Q)$ is defined as

$$\mathbf{A} = \begin{bmatrix} a_0^{(1)} & a_0^{(2)} & \cdots & a_0^{(Q)} \\ a_1^{(1)} & a_1^{(2)} & \cdots & a_1^{(Q)} \\ \vdots & \vdots & & \vdots \\ a_m^{(1)} & a_m^{(2)} & \cdots & a_m^{(Q)} \end{bmatrix}. \qquad (2)$$

Modes have been used by the signal processing community for a long time, albeit under a different definition. Both definitions are equivalent for most applications. Definition 2.1 leads to simple formulations of the relations between modes and polynomials, such as the fundamental property exposed by the following lemma.

Lemma 2.1. Given a set of polynomials $\mathcal{A} = \left\{ a_i(x) \;\middle|\; \deg(a_i(x)) \le m,\ i = 1,\ldots,Q \right\}$ with coefficients matrix $\mathbf{A} \in \mathcal{M}_{\mathbb{C}}(m+1,Q)$, a set of associate modes of a polynomial is included in the orthogonal complement of $\mathrm{Range}(\mathbf{A})$ if and only if they are complex conjugates of the associate modes of the GCD of the set of polynomials $\mathcal{A}$.

2.2. Constrained optimization problem

We follow the definition of the d-AGCD to pose a constrained optimization problem. The distance between given and modified polynomial sets requires a suitable definition of a polynomial metric. We adopt the sum of squared differences between the coefficients of corresponding polynomials in both sets, which leads to a least squares estimation of the GCD. It can be conveniently expressed as the squared Frobenius norm of the difference between the coefficients matrices of the input set, $\mathbf{A}$, and the modified one, $\hat{\mathbf{A}}$: $\left\|\mathbf{A} - \hat{\mathbf{A}}\right\|_F^2$.

Lemma 2.1 gives the condition that must be imposed on $\hat{\mathbf{A}}$ so that the modified polynomial set has a d-degree GCD. Therefore, the AGCD of a set of polynomials with coefficients matrix $\mathbf{A}$ can be computed by solving the following conditioned nonlinear optimization problem:

$$\begin{aligned} &\min_{\{\hat{\mathbf{A}} \in \mathcal{M}_{\mathbb{C}}(m+1,Q)\}} \left\|\mathbf{A} - \hat{\mathbf{A}}\right\|_F^2 \\ &\text{subject to:} \quad \mathbf{T}^{\mathrm{H}} \cdot \hat{\mathbf{A}} = \mathbf{0} \\ &\qquad\qquad\quad \mathbf{T}^{\mathrm{H}} \cdot \mathbf{T} = \mathbf{I}_d \\ &\qquad\qquad\quad \mathrm{Range}(\mathbf{T}) \text{ spanned by the associate modes} \\ &\qquad\qquad\qquad \text{of a } d-\text{degree polynomial} \end{aligned} \qquad (3)$$

The coefficients of the AGCD are the complex conjugates of those of the polynomial $b(x)$ whose associate modes span $\mathrm{Range}(\mathbf{T})$. Problem (3) is no longer ill-posed, but still difficult to solve due to its conditioned nature.

The objective function in (3) can be simplified and matrix $\hat{\mathbf{A}}$ removed from the optimization problem. Let the set of columns of $\mathbf{T}$ be extended to a base of $\mathbb{C}^{(m+1)}$, so that $[\mathbf{T} \quad \mathbf{S}]$ is an orthogonal matrix. Then, we have:

$$\left\|\mathbf{A} - \hat{\mathbf{A}}\right\|_F^2 = \left\|\mathbf{T}^{\mathrm{H}} \cdot \left(\mathbf{A} - \hat{\mathbf{A}}\right)\right\|_F^2 + \left\|\mathbf{S}^{\mathrm{H}} \cdot \left(\mathbf{A} - \hat{\mathbf{A}}\right)\right\|_F^2 \qquad (4)$$

Under condition $\mathbf{T}^{\mathrm{H}} \cdot \hat{\mathbf{A}} = \mathbf{0}$, the objective function is

$$\left\|\mathbf{A} - \hat{\mathbf{A}}\right\|_F^2 = \left\|\mathbf{T}^{\mathrm{H}} \cdot \mathbf{A}\right\|_F^2 + \left\|\mathbf{S}^{\mathrm{H}} \cdot \left(\mathbf{A} - \hat{\mathbf{A}}\right)\right\|_F^2. \qquad (5)$$

Clearly, the minimum of the objective function under the constraints in (3) will be attained for a matrix $\hat{\mathbf{A}}$ with columns in $\mathrm{Range}(\mathbf{S})$ such that $\mathbf{S}^{\mathrm{H}} \cdot \left(\mathbf{A} - \hat{\mathbf{A}}\right) = \mathbf{0}$. That is, the columns of $\hat{\mathbf{A}}$ must be equal to the projections of the columns of $\mathbf{A}$ onto $\mathrm{Range}(\mathbf{S})$. The estimated AGCD is completely determined by $\mathbf{T}$. Instead of the objective function in (3) we can equivalently minimize

$$\left\|\mathbf{T}^{\mathrm{H}} \cdot \mathbf{A}\right\|_F^2 = \mathrm{tr}\left(\mathbf{T}^{\mathrm{H}} \cdot \mathbf{A}\mathbf{A}^{\mathbf{H}} \cdot \mathbf{T}\right) = \sum_{k=1}^{d} \mathbf{t}_k^{\mathrm{H}} \cdot \mathbf{A}\mathbf{A}^{\mathbf{H}} \cdot \mathbf{t}_k. \qquad (6)$$

Now, the last condition in (3) will be expressed in matrix form. To this end, the following lemma is introduced.

Lemma 2.2. Given a d-degree polynomial $b(x) = b_0 + b_1 \cdot x + b_2 \cdot x^2 + \ldots + b_d \cdot x^d$, and a Toeplitz matrix $\mathbf{B} \in \mathcal{M}_{\mathbb{C}}(m+1, m-d+1)$, $d \le m$, defined as

$$\mathbf{B}^{\mathrm{T}} = \begin{bmatrix} b_0 & b_1 & \cdots & b_d & & \\ & \ddots & \ddots & & \ddots & \\ & & b_0 & b_1 & \cdots & b_d \end{bmatrix}, \qquad (7)$$

the orthogonal complement subspace of $\mathrm{Range}(\mathbf{B})$ is spanned by the conjugates of the associate modes of $b(x)$.

Lemma 2.2 gives the relation that must hold if $\mathrm{Range}(\mathbf{T})$ is spanned by the conjugates of the associate modes of a polynomial $b(x)$: $\mathbf{B}^{\mathrm{H}} \cdot \mathbf{T} = \mathbf{0}$. As $b(x)$ is unique up to scalar multiplication, some sort of normalization is needed. We require its vector of coefficients to have unit norm, giving the following equivalent optimization problem:

$$\begin{aligned} &\min_{\{\mathbf{b} \in \mathcal{M}_{\mathbb{C}}(d+1,1)\}} \mathrm{tr}\left(\mathbf{T}^{\mathrm{H}} \cdot \mathbf{A}\mathbf{A}^{\mathrm{H}} \cdot \mathbf{T}\right) \\ &\text{subject to:} \quad \mathbf{B}^{\mathrm{H}} \cdot \mathbf{T} = \mathbf{0} \\ &\qquad\qquad\quad \mathbf{T}^{\mathrm{H}} \cdot \mathbf{T} = \mathbf{I}_d \\ &\qquad\qquad\quad \mathbf{b}^{\mathrm{H}} \cdot \mathbf{b} = 1 \end{aligned} \qquad (8)$$

2.3. Sequential unconstrained optimization problem

Linearizing the conditions in (8) around a pair of matrices $\mathbf{B}$, $\mathbf{T}$ satisfying them yields the following expressions:

$$\begin{aligned} \Delta\mathbf{B}^{\mathrm{H}} \cdot \mathbf{T} + \mathbf{B}^{\mathrm{H}} \cdot \Delta\mathbf{T} &= \mathbf{0} \\ \Delta\mathbf{T}^{\mathrm{H}} \cdot \mathbf{T} + \mathbf{T}^{\mathrm{H}} \cdot \Delta\mathbf{T} &= \mathbf{0} \\ \Delta\mathbf{b}^{\mathrm{H}} \cdot \mathbf{b} + \mathbf{b}^{\mathrm{H}} \cdot \Delta\mathbf{b} &= 0 \end{aligned} \qquad (9)$$

The second and third equations in (9) hold if the columns of $\Delta\mathbf{T}$ and $\Delta\mathbf{b}$ are restricted to be orthogonal to the columns of $\mathbf{T}$ and $\mathbf{b}$, respectively. The minimum norm solution for $\Delta\mathbf{T}$ in the first equation in (9) lies in $\text{Range}(\mathbf{B})$, since components in its orthogonal complement, $\text{Range}(\mathbf{T})$, do not contribute to the right side. So the solution to the first two equations in (9) is given by

$$\Delta\mathbf{T} = -\left(\mathbf{B}^{\mathrm{H}}\right)^{+} \cdot \Delta\mathbf{B}^{\mathrm{H}} \cdot \mathbf{T}, \tag{10}$$

Furthermore, due to the structure of matrix $\Delta\mathbf{B}$, (10) can be transformed into

$$\Delta\mathbf{T} = -\left(\mathbf{B}^{\mathrm{H}}\right)^{+} \cdot \left[\mathbf{L}_1 \cdot \Delta\mathbf{b}^* \quad \mathbf{L}_2 \cdot \Delta\mathbf{b}^* \quad \cdots \quad \mathbf{L}_d \cdot \Delta\mathbf{b}^*\right], \tag{11}$$

with $\mathbf{L}_k \in \mathcal{M}_{\mathbb{C}}(m-d+1, d+1)$, $k = 1,\ldots,d$, defined as:

$$\mathbf{L}_k = \begin{bmatrix} t_{1,k} & t_{2,k} & \cdots & t_{(d+1),k} \\ t_{2,k} & t_{3,k} & \cdots & t_{(d+2),k} \\ \vdots & \vdots & & \vdots \\ t_{(m-d+1),k} & t_{(m-d+2),k} & \cdots & t_{(m+1),k} \end{bmatrix}. \tag{12}$$

Let $\bar{\mathbf{t}} \in \mathcal{M}_{\mathbb{R}}(2d \cdot (m+1), 1)$ be a real column vector built with the real and imaginary parts of the columns of $\mathbf{T}$:

$$\bar{\mathbf{t}}^{\mathrm{T}} = \left[\mathbf{t}_1^{(r)} \quad \mathbf{t}_1^{(i)} \quad \cdots \quad \mathbf{t}_d^{(r)} \quad \mathbf{t}_d^{(i)}\right]. \tag{13}$$

Equation (11) allows to express parameter vector $\Delta\bar{\mathbf{t}}$ in terms of $\Delta\mathbf{b}$. To this end, we define matrices $\mathbf{C}_k \in \mathcal{M}_{\mathbb{C}}(m+1, d+1)$, $k = 1,\ldots,d$, as $\mathbf{C}_k = \left(\mathbf{B}^{\mathrm{H}}\right)^{+} \cdot \mathbf{L}_k$ and note that, from (11), we have

$$\Delta\bar{\mathbf{t}} = \begin{bmatrix} \Delta\mathbf{t}_1^{(r)} \\ \Delta\mathbf{t}_1^{(i)} \\ \vdots \\ \Delta\mathbf{t}_d^{(r)} \\ \Delta\mathbf{t}_d^{(i)} \end{bmatrix} = \mathbf{V} \cdot \begin{bmatrix} \Delta\mathbf{b}^{(r)} \\ \Delta\mathbf{b}^{(i)} \end{bmatrix}, \tag{14}$$

with

$$\mathbf{V} = \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \vdots \\ \mathbf{E}_d \end{bmatrix}, \quad \mathbf{E}_k = \begin{bmatrix} -\mathbf{C}_k^{(r)} & -\mathbf{C}_k^{(i)} \\ -\mathbf{C}_k^{(i)} & \mathbf{C}_k^{(r)} \end{bmatrix}, \quad k = 1,\ldots,d. \tag{15}$$

Finally, the orthogonality condition $\mathbf{b}^{\mathrm{H}} \cdot \Delta\mathbf{b} = \mathbf{0}$ can be expressed in terms of the real and imaginary parts of the vectors involved as:

$$\begin{bmatrix} \mathbf{b}^{(r)\mathrm{T}} & \mathbf{b}^{(i)\mathrm{T}} \\ -\mathbf{b}^{(i)\mathrm{T}} & \mathbf{b}^{(r)\mathrm{T}} \end{bmatrix} \cdot \begin{bmatrix} \Delta\mathbf{b}^{(r)} \\ \Delta\mathbf{b}^{(i)} \end{bmatrix} = \mathbf{0} \tag{16}$$

Let $\mathbf{Q} \in \mathcal{M}_{\mathbb{R}}(2(m+1), 2(m-1))$ be a full column rank matrix so that $\text{Range}(\mathbf{Q})$ is the right null space of the first matrix in (16). Then we have

$$\begin{bmatrix} \Delta\mathbf{b}^{(r)} \\ \Delta\mathbf{b}^{(i)} \end{bmatrix} = \mathbf{Q} \cdot \mathbf{x} \tag{17}$$

for some column vector $\mathbf{x} \in \mathcal{M}_{\mathbb{R}}(2(m-1), 1)$.

Starting from a pair $\mathbf{B}$, $\mathbf{T}$ that satisfy the conditions of the optimization problem (8), the perturbation that can be applied to their elements so that the conditions are still met is given by (14) and (17). Incorporating these results to the Taylor expansion of the objective function $f(\bar{\mathbf{t}})$ in (8),

$$f(\bar{\mathbf{t}} + \Delta\bar{\mathbf{t}}) = f(\bar{\mathbf{t}}) + \left(\nabla_{\bar{\mathbf{t}}} f\right)^{\mathrm{T}} \cdot \Delta\bar{\mathbf{t}} + \frac{1}{2} \cdot \Delta\bar{\mathbf{t}}^{\mathrm{T}} \cdot \nabla_{\bar{\mathbf{t}}}^2 f \cdot \Delta\bar{\mathbf{t}}, \tag{18}$$

yields the objective function of the equivalent unconstrained optimization problem:

$$\begin{aligned} f(\bar{\mathbf{t}}, \mathbf{x}) = {} & f(\bar{\mathbf{t}}) + \left(\nabla_{\bar{\mathbf{t}}} f\right)^{\mathrm{T}} \cdot \mathbf{V} \cdot \mathbf{Q} \cdot \mathbf{x} + \\ & \frac{1}{2} \cdot \mathbf{x}^{\mathrm{T}} \cdot \mathbf{Q}^{\mathrm{T}} \cdot \mathbf{V}^{\mathrm{T}} \cdot \nabla_{\bar{\mathbf{t}}}^2 f \cdot \mathbf{V} \cdot \mathbf{Q} \cdot \mathbf{x} \end{aligned} \tag{19}$$

The AGCD computation method we propose consist of the following steps:

1. Initialization: Find initial values for $\mathbf{b}$ and $\mathbf{T}$. Methods based of the Sylvester matrix of the set of polynomials give good initial values.
2. Compute vector $\bar{\mathbf{t}}$ (9), matrices $\mathbf{V}$ (15), $\mathbf{Q}$ (17).
3. Find the value of vector $\mathbf{x}$ that minimizes (19).
4. Update $\mathbf{b}$ (17), and find $\mathbf{T}$ such that $\mathbf{B}^{\mathrm{H}} \cdot \mathbf{T} = \mathbf{0}$.
5. If convergence has not been reached, go to step 2.

The coefficients of the AGCD are given by the complex conjugates of the elements of $\mathbf{b}$.

3. PERFORMANCE TEST

The accuracy of the estimation provided by the proposed method has been compared with the results obtained with other methods. To this end, the common factor estimation method (COFE) [10], the resultant matrix pencil method (Res-MP) [13] and the proposed approach were implemented in MATLAB. A benchmark example was selected from [14]. It consists of a set $\mathcal{A}$ of 8 polynomials of degree 20 with GCD of degree 3:

$$\begin{aligned} \mathcal{A} = {} & \{a_i(x) = g(x) \cdot u_i(x), i = 1,\ldots,8\} \\ g(x) = {} & 2x + 4x + 3x^2 + x^3 \\ u_1(x) = {} & 70 + 19x + 2x^3 + 5x^6 + x^9 + 11x^{10} + \\ & + x^{12} + 7x^{13} + 15x^{14} + x^{16} + x^{17} \\ u_2(x) = {} & x^3 + 2x^5 + 5x^7 + 3x^{11} + x^{17} \\ u_3(x) = {} & 11x + x^3 + x^{10} + x^{17} \\ u_4(x) = {} & 10 + 2x^5 + 4x^{10} + 5x^{17} \\ u_5(x) = {} & 30 - x^5 - x^{17} \\ u_6(x) = {} & 11 - 3x^6 + x^9 - 2x^{13} + x^{17} \\ u_7(x) = {} & 20 + 11x^7 + x^{17} \\ u_8(x) = {} & 9 + 5x^6 + 2x^{12} + 3x^{15} + x^{17} \end{aligned} \tag{20}$$

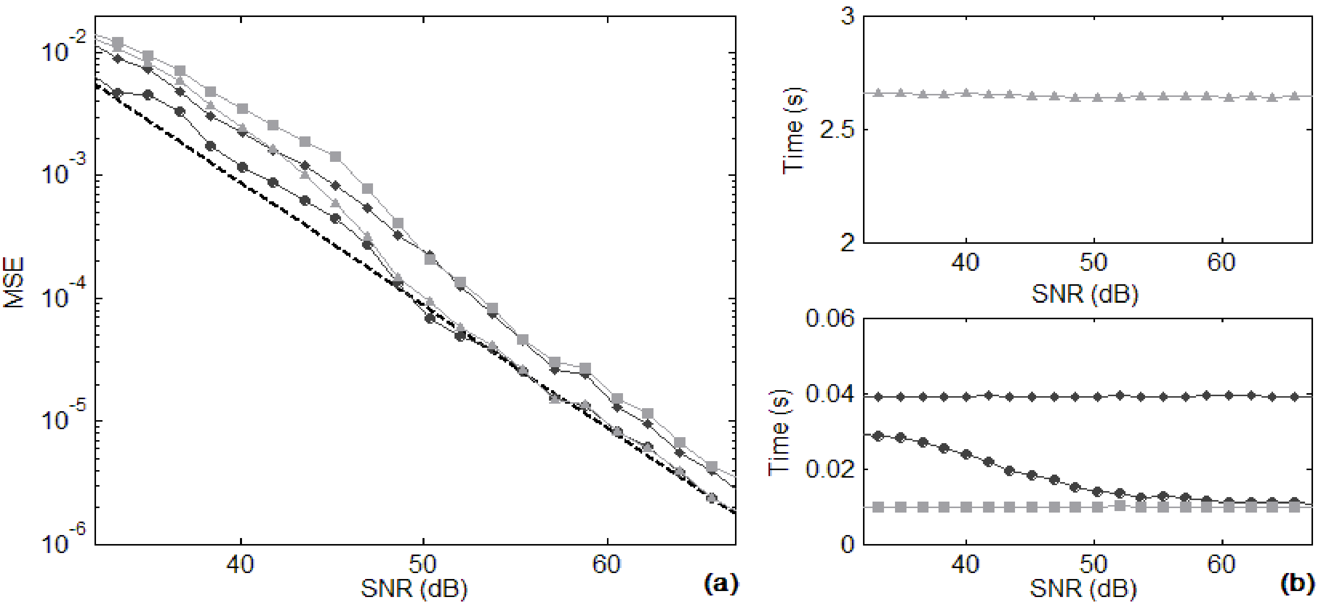

Fig. 1. (a): CRB (dashed) and MSE of AGCD computation methods: COFE (0 iterations) (diamonds), COFE (2 iterations) (triangles), Resultant MP (squares) and proposed method (circles). (b): Mean computation time for each method implemented in MATLAB.

Zero-mean Gaussian noise was added to the coefficients of the polynomials in $\mathcal{A}$. Different levels of noise, leading to SNR values between 15 and 72, were considered. For each noise level, 100 realizations were generated and the AGCD estimated with each method.

Fig. 1 (a) shows the MSE for each method, along with the Cramer-Rao lower bound (CRB) computed for this example. Both the proposed method and COFE with 2 iterations attain the CRB for SNR values above 48 dB. However, due to the lower number of parameters and the smaller size of the matrices involved, the proposed approach is much faster, as Fig. 1 (b) shows. This is an important feature, since high computational burden may impair the application of a GCD computation method to polynomials of large degree.

Only Res-MP is faster then the proposed method for low SNR values, but it shows lower accuracy in the estimations and fails to attain the CRB for high SNR values.

4. APPLICATION EXAMPLE

This section shows an application of AGCD computation to blind deconvolution in single input multiple output (SIMO) linear time invariant (LTI) systems. One such system, with three channels, is depicted if Fig. 2.

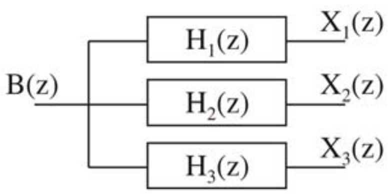

Fig. 2. SIMO system.

Under the LTI assumption, the Z-transforms of the outputs are equal to the product of the Z-transforms of the common input and the channels:

$$X_k(z) = B(z) \cdot H_k(z), \quad k = 1,2,3. \tag{21}$$

If all signals involved have finite length, then the Z-transforms are finite-degree polynomials in z^{-1} whose coefficients are the samples of the signals. Furthermore, if the Z-transforms of the channels do not have a zero common to all of them, then $B(z)$ is the GCD of the set $\{X_1(z), X_2(z), X_3(z)\}$. Thus, through a GCD computation, the input signal can be estimated blindly from the outputs of the system, that is, without knowledge about the channels. This approach requires a robust AGCD computation method to cope with measurement and model errors. The suitability of the proposed method for this task has been tested with a force's time-history recovery experiment.

The experimental set-up for the acquisition of real signals consists of a freely supported steel beam hit by a sensorized hammer (Fig. 3). The acceleration caused by the impact is detected by four piezoelectric accelerometers placed upon that beam. These sensors provide the output of the multichannel system.

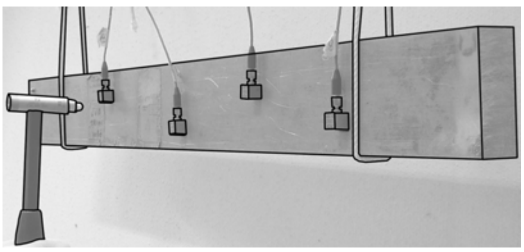

Fig. 3. Experimental set-up: Freely supported beam, accelerometers and impact hammer.

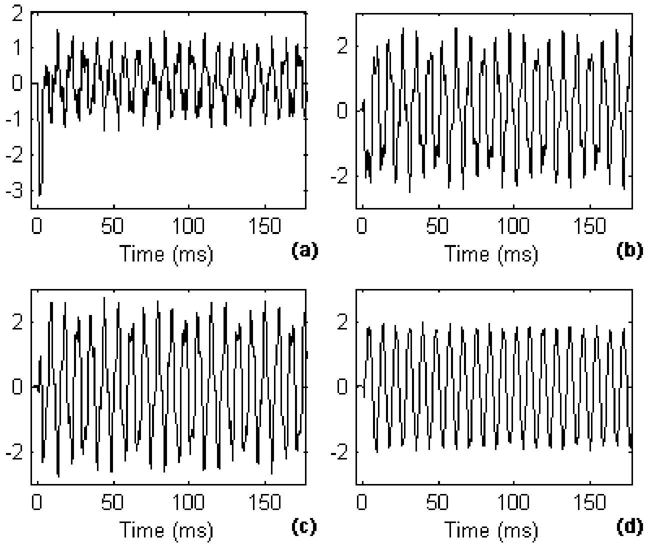

Fig. 4. Output signals from accelerometers.

Figure 4 shows the four output signals. The accelerometer in the head of the hammer gives a direct measure of the excitation signal. It is not used by the deconvolution method; it only serves as a reference to assess the accuracy of the estimation. All signals are sampled at 4096 samples/second by the acquisition device.

The infinite length of the output signals, due to the excitation of vibration modes, violates the assumption of finite-length signals. In order to apply the proposed algorithm, some processing must be done prior to the GCD computation. The channels transforming the impact signal into the accelerations sensed by the accelerometers can be regarded as infinite impulse response (IIR) systems. Their system response is:

$$H_k(z) = \frac{N_k(z)}{P(z)}, \quad k = 1,2,3,4. \tag{22}$$

The denominator $P(z)$ is common to all of them. Its roots give the frequency and damping of the vibration modes, which depend on the characteristics of the beam and not on the position of the impact and measurement points.

If the poles are estimated from the output signals, an estimation $\widehat{P}(z)$ of the denominator can be obtained. Then, a finite impulse response (FIR) filter with system response $\widehat{P}(z)$ can be applied to the output signals to transform them into finite-length signals:

$$X_k(z) \cdot \widehat{P}(z) = B(z) \cdot \frac{N_k(z)}{P(z)} \cdot \widehat{P}(z) \approx B(z) \cdot N_k(z), \quad k = 1,\ldots,4. \tag{23}$$

There are many methods proposed in the literature for solving the so-called exponential modelling problem of estimating the parameters of damped sinusoids. We have used the approach described in [15]. Fig. 5 shows the estimated poles in the Z-plane.

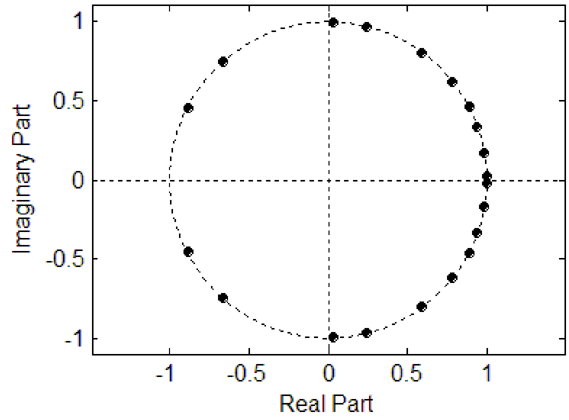

Fig. 5. System poles estimated from output signals.

The results of filtering the output signals with a FIR filter whose system response has the poles of Fig. 4 as zeros is shown in Fig. 6. These are finite-length signals that come from the same input, which can be obtained through GCD computation, as explained at the beginning of this section.

The proposed d-AGCD computation method is applied to the finite-length signals. This method requires a value for the degree of the GCD. Several methods to estimate it have been published [5],[8]. However, we have adopted a simpler approach: The approximate GCD is computed for several degrees, and the one that gives a backward error significantly smaller than the next one is chosen as the candidate GCD degree.

Fig. 7 shows the impact signal provided by the sensor embedded in the hammer (dotted line) and the estimated one, provided by the proposed method (solid line). Note that the estimation has been conveniently scaled, since this information cannot be obtained through blind deconvolution. The estimation matches closely the reference signal, and parameters such as signal length, rise and decay rate could be computed from the estimation.

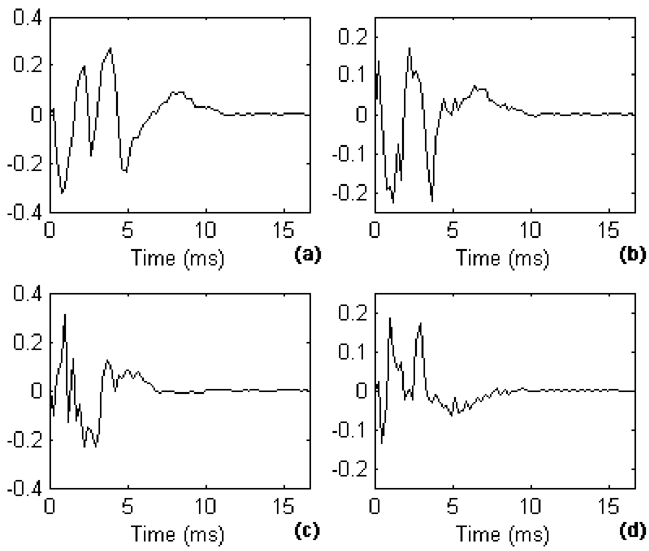

Fig. 6. Output signals after removing poles' contribution.

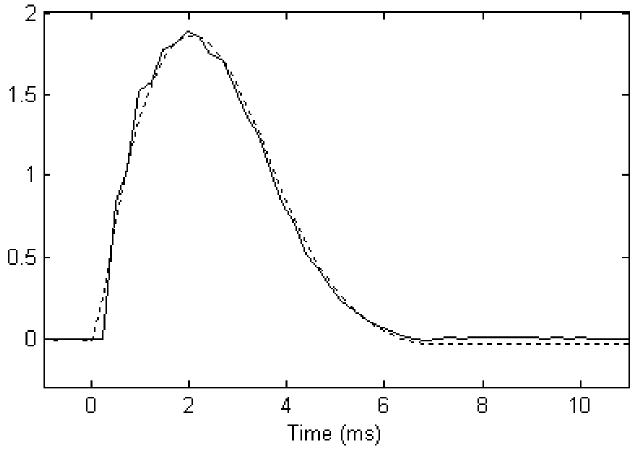

Fig. 7. Estimated input signal (solid line) and reference from impact hammer (dotted line).

4. CONCLUSIONS

This paper gives a full description of a novel AGCD computation method based on the relation of modes and polynomials. The proposed algorithm consists of sequential minimizations of quadratic functions, hence its fast performance and ability to work with sets of many polynomials of large degree. This feature allows its application to many measurement techniques involving polynomial models. It compares favourably with other AGCD computation methods. Thorough comparisons can be found in [12].

An example is given that shows how the proposed AGCD computation method can be used to perform blind deconvolution of finite signals in a SIMO framework. The time history of the force exerted by a hammer on a beam has been recovered (up to a scaling factor) from the acceleration signals provided by sensors placed on the beam's surface.

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