XIX IMEKO World Congress
Fundamental and Applied Metrology
September 6–11, 2009, Lisbon, Portugal

GRADUALLY-VARIED FLOWS IN OPEN-CHANNEL NETWORKS

Maria Teresa Todisco

Structures, Waters and Land Engineering Department-
Faculty of Engineering – L'Aquila, Italy. mariateresa.todisco@univaq.it; mt.todisco@gmail.com

Abstract – Reported here are some results obtained, from a calculus algorithm application that, based on known equation analytical solutions of water flow profiles, allows the study of steady-state, gradually varied flow in open channel networks. The procedure allows to calculate, in the case of slow water flow into gradually downward slope channels, in the direction of motion, the florates and water levels respectively, in all the nodes and sides of the network. The results have been compared with the solutions proposed by other authors.

Keywords: open-channel networks gradually-varied flows

1. THEORETICAL MODEL

The hydraulic calculation to verify and design the open channel networks are usually based on the hypothesis that the water discharge of each network element diffuses in constant motion conditions.

This hypothesis doesn't really conform to the real liquid movement conditions, rather a more adhering evaluation of the hydraulic reality phenomenon established in the networks which should foresee going back to the varied motion equations.

For incompressible liquids these latter are reduced to continuous movement equations. Together they are a differential equation system to the partial derivates that, excluding very few particular cases, defined by drastic simplifications with respect to real phenomenon, don't give finite term solutions.

The difficulties met in the water flow study in unsteady-state, are found not only in the analytical problem to prevent solutions of word equations, but mostly in defining proper limit and initial conditions.

Initial conditions, in particular, are defined by steady-state situations existing before the varied regimes takes place: then, for their definition, it's necessary to point out the discharges and the water surface in all network elements.

The definition of these latter, in some conditioning hypotheses on which we'll return to in the future, has been obtained in [1].

The algorithm proposed is based on known analytical solutions of the water flow profile obtained by [2] and by [3].

The physical model is made up of an open channel network in which diffuse assigned discharges (measurable through instruments) and where the boundary conditions will be the water levels recorded at the extreme channel ends.

2. WATER SURFACE PROFILES TRACING

The resolving procedures leading to steady-state flow profile tracing, gradually varied, flowing at a constant Q flowrate in cylindrical channels, are seen in the integration of the differential equation [4]:

$$\frac{dh}{ds} = \frac{i_f - J(h)}{i - \Omega(h)} \tag{1}$$

The hydraulic calculations to verify and design open channel networks are usually based on (1) being: h flow depth in a generic section, i_f slope of channel bottom (constant); $J(h)$ energy dissipation (per weight and path unit) due to the sole resistance of friction; $\Omega(h) = \alpha Q^2 l / g\sigma^3$; α the Coriolis co-efficient (later considered constant and equal to 1); l the width of water section of height h; g acceleration of gravity; s measured distance along channel access from the section taken as the origin (positive in the direction of movement).

The equation solution in finite form (1) can be obtained only for channels in which the transversal sections is of such form that between areas σ and corresponding heights h the monomials relation exists

$$\sigma = bh^n \tag{2}$$

In practical terms (2) is satisfactory due to the almost completeness of the sections used in the open horizon channels within the approximation limits normally allowed in such kind of a problem: the only exceptions are given by the closed limit sections used in the underground channelling [3], [4].

The law of resistance, generally expressed by the relation of the type:

$$J = f(\psi, R, S, \rho, \mu, \upsilon) \tag{3}$$

where ψ is one or more parameters characterising the channel section shape, R the average radius, S one or more parameters characterising the wall roughness, ρ and μ liquid density and viscosity, v the average flow velocity – can be specialised in the monomial relation :

$$Q = kh^p J^q \tag{4}$$

where k indicates a dimensional parameter, essentially wall-natured and parameter dependant, defining the channel transversal section form. Under these conditions, the differential equation (1) can be integrated [2],[4].

Considering (2) and (4), (1) becomes:

$$\frac{dh}{ds} = i_f \frac{1 - \left(\frac{Q}{i_f^q k}\right)^{1/q} \frac{1}{h^{p/q}}}{1 - \frac{nQ^2}{gb^2}\frac{1}{h^{2n+1}}} \quad (5)$$

where the terms:

$$h_u = \left(\frac{Q}{i_j^q k}\right)^{(1/p)} \quad \text{and} \quad h_c = \left(\frac{nQ^2}{gb^2}\right)^{1/(2n+1)}$$

represent, respectively, height h_u of uniform movement and height h_c of critical state, both related to discharge Q.

Replacing these expressions in (5), separating variables and imposing: $z = \frac{h}{h_u}$ [1] is obtained, general integral of (5):

where z<1

$$s = \frac{h_u}{i_f}\left[z - \varphi_{\frac{p}{q}}(z) + \frac{q}{p-2nq}\left(\frac{h_c}{h_u}\right)^{2n+1} \varphi_{\frac{p}{p-2nq}}\left(z^{\frac{p}{p-2nq}}\right)\right] + c \quad (6)$$

where z>1

$$s = \frac{h_u}{i_f}\left[z - \varphi_{\frac{p}{q}}(\theta) + \frac{q}{p-2nq}\left(\frac{h_c}{h_u}\right)^{2n+1} \varphi_{\frac{p}{p-2nq}}\left(\theta^{\frac{p}{p-2nq}}\right)\right] + c \quad (7)$$

where constant $c = c_1+c_2$ defined by limit conditions.

Equations (6) and (7) deduced from [4] through non-quadratic resistance laws defined in (5) are particular in solution [2] when q=1/2, obtaining:

where z<1

$$s = \frac{h_u}{i_f} F\left(\frac{h}{h_u}\right) + c \quad (8)$$

being:

$$F\left(\frac{h}{h_u}\right) = \frac{h}{h_u} - \varphi_{2p}\left(\frac{h}{h_u}\right) + \frac{1}{2(p-n)}\left(\frac{h_c}{h_u}\right)^{2n+1} \varphi_{\frac{p}{p-n}}\left(\left(\frac{h_c}{h_u}\right)^{2n+1}\right) \quad (9)$$

where z>1:

$$s = \frac{h_u}{i_f} G\left(\frac{h_u}{h}\right) + c \quad (10)$$

being:

$$G\left(\frac{h_u}{h}\right) = \frac{h}{h_u} - \varphi_{p/q}\left(\frac{h_u}{h}\right) + \frac{q}{p-2nq}\left(\frac{h_c}{h_u}\right)^{2n+1} \varphi_{\frac{p}{p-2nq}}\left(\left(\frac{h_u}{h}\right)^{p-2nq/q}\right) \quad (11).$$

When discharge Q is assigned in considered channel section, distance l separating two water sections in which known water levels values $h_1 = z_1 h_u$ e $h_2 = z_2 h_u$ (linked by a continuous water profile), is given by:

for z<1

$$l = s_2 - s_1 = \frac{h_u}{i_f}\left[F\left(\frac{h_2}{h_u}\right) - F\left(\frac{h_1}{h_u}\right)\right] \quad (12)$$

for z>1

$$l = s_2 - s_1 = \frac{h_u}{i_f}\left[G\left(\frac{h_u}{h_2}\right) - G\left(\frac{h_u}{h_1}\right)\right] \quad (13).$$

In general, 12 and 13 can be placed under generic form:

$$l = \frac{h_u}{i_f}\Gamma[z_1(h_u), z_2(h_u), h_u] \quad (14)$$

being:

$$\Gamma(h_1, h_2, h_u) = F\left(\frac{h_2}{h_u}\right) - F\left(\frac{h_1}{h_u}\right) \quad (15)$$

for z<1

$$\Gamma(h_1, h_2, h_u) = G\left(\frac{h_u}{h_2}\right) - G\left(\frac{h_u}{h_1}\right) \quad (16)$$

for z>1.

Eq. (14) can be used both for obtaining water levels h_2(or h_1) when Q, l, h_1(oh_2) are assigned, and for calculating regular regime depth h_u and, then, discharge Q – when l, h_1 and h_2 are assigned: we'll have to obtain equation solution:

$$\Phi[z_1(h_u), z_2(h_u)] = 0 \quad (17)$$

being

$$\Phi[z_1(h_u), z_2(h_u)] = 1 - \frac{h_u}{i_f l}\Gamma[z_1(h_u), z_2(h_u)] \quad (18).$$

3. CHANNEL NETWORKS

Eq.(14) allows to easily calculate distance l separating two water levels h_1 and h_2 when the discharge is known. Moreover, It allows the determination of discharge Q in a cylindrical channel when water levels h_{1*} and h_{2*} are assigned at the extreme channel endings, together whit the separating distance l.

This is the start for perfecting the calculus algorithm in verifying channel network.

For a general model, a channel network can be considered made up of $\tilde{l}$ sides (channel sections of regular roughness form) and $\tilde{n}$ nodal points (points where three or more channels converge, or points of the same channel where there's a sudden change of section, wall roughness, bottom, slope, discharge, etc.)

We'll call: side borders, sides to which are assigned, at one end, at least a border condition (water level value, or allowed or derogated discharge) and *internal sides*, the remaining; the same distinction's true for nodes.

We'll call: *walk or path*, any open polygon made up of consecutive sides belonging to branching, having an end in a border node, and the other end in any other node (internal or border).

For each side an initial and final node's defined and, then, a conventional flow pattern.

Without regulating and measuring devices, expanding basins, mechanical relief implantations, until steady-state motion is completely known, it's necessary to know piezometric values of each $\tilde{n}$ node and discharge in $\tilde{l}$ sides.

If $\tilde{n}' \leq \tilde{n}$ a number ofis unknown piezometric values and $\tilde{l}' \leq \tilde{l}$ of unknown discharge, the corresponding mathematical model's:

$\tilde{l}$ movement equations (type eq. (14))

$\tilde{n}$ ' nodal equations.

These latter are reduced to the equation expressing discharge continuity for each node, and to these are associated other defining conditions to define according to flow type (sub critical or supercritical) and to node typology.

The extensive variety of possible situations doesn't allow for an effective topic synthesis.

Here, we pay attention to slow discharge flows, gradually varied, in open network channels, in the direction of the motion; the definition of the condition to associate to the continuity equation has been obtained assuming that on the nodal points constituted from the union of two or more channels, the kinetic heights merely differ, and local node charge loss needn't be considered. [5]

Such hypotheses are specialised according to the situation drafted in fig. 1a in the absence of allowed or externally derogated discharge, in the relations:

$$Q_j - Q_{j+1} - Q_{j+2} = 0 \qquad (19a)$$

$$y_{j,k} - y_{j+1,k} = 0 \qquad (19b)$$

$$y_{j,k} - y_{j+2,k} = 0 \qquad (19c)$$

and for situation drafted in fig.1b. in the relations:

$$Q_{j+1} + Q_{j+2} - Q_{j+3} = 0 \qquad (20a)$$

$$y_{j+1,k} - y_{j+3,k} = 0 \qquad (20b)$$

$$y_{j+2,k} - y_{j+3,k} = 0 \qquad (20c)$$

having indicated with:

$y_{i,k} = h_{j,k} + z_{f_{j,k}}$ and with $z_{f_{j,k}}$ (and analogous), respectively, the piezometric value and the bottom value with respect to a horizontal reference in node k believed to belong to side j.

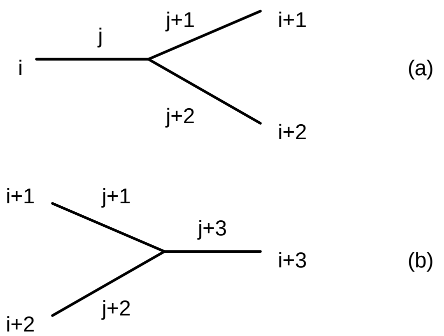

Fig.1

For an open network characterised by $\tilde{n}' \leq \tilde{n}$ nodes of unknown piezometric value, one obtains a system formed by $\tilde{n}'$ equations expressing the continuity of nodal discharges.

The solution of this system can be obtained with iteration procedures.

Therefore, even when in all the network sides, the necessary conditions are found for gradual movement in the direction of decreasing bottom values, application of these procedures can cause operational difficulty.

To reduce such difficulty one must observe that various network nodes depend among themselves, since not all unknown $\tilde{n}'$ water heights can be randomly assigned; until gradual movement in all network sides is respected it's necessary that unknown water heights values are within interval (h_{min}, h_{max}) depending on the sides' geometric and hydraulic characteristics, and through conditions given in external nodes.

In pointing out this last question, network side j with node I border end is examined, for example, and other end in internal node k (fig.2).

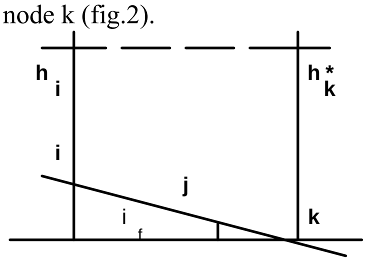

Fig.2

In side j discharge $Q_j \neq 0$ for major and minor h_k^* values, being $h_k^* = h_i + i_{f_j} l_j$ hk=hi+il.

Conditions $h_k > h_k^*$ and $h_k < h_k^*$ are respectively relative to delayed slow flows in channels with a deep slope, and to delayed slow flows or accelerated flow, in channels having a slight slope.

In discerning flow type assume $h_i = h_c$; from relation $h_c = \left[nQ_j^2 / g b_j^2 \right]^{1/(2n_j+1)}$, Q_j value's seen and then the value of $h_{u_j} = \left[Q_j / \left(k_j i_j^q \right) \right]^{1/p_j}$.

If $h_c > h_u$ (deep slope channel) max value $h_{k_{max}} > h_k^*$ assumed by water level under imposed conditions, it holds introducing in (14), that Q_j, l_j, h_i. Min value $h_{k_{min}} = h_u$. Here h_k variation interval is between [h_u,h_{kmax}].

If $h_c < h_u$ (slight slope channel), min value $h_{k_{min}}$ assumed by unknown water level under assigned conditions, a critical state condition is determined, imposing in k, or rather searching for value $h_k < h_i$, giving max compatible flow under imposed conditions. Max value's determined for h_k^*. h_k variation interval results between [h_{kmin},h_k].

One works with the same procedure when I's the internal node and k the external one, or when in internal node k converge external sides r. Here, in each side $\left(h_{k\ \min}^{j}, h_{k\ \max}^{j}\right)$ intervals are defined, with $j \in \{1, r\}$ where:

$$h_{k_{\min}} = \max\left\{h_{k\ \min}^{j}\right\}$$
$$h_{k_{\max}} = \min\left\{h_{k\ \max}^{j}\right\} .$$

It's a different situation for internal nodes linked to others belonging to border sides. In general for these it's not possible to find one interval in which to continuously vary water level values. Then, once a first position value is assigned in a chosen internal node of a border side, water level values in other internal nodes externally linked are defined.

In pointing out such a question a channel network's examined (fig.3), made up of concurring branches in node D linked to border H, in correspondence to which water level $\bar{h}_H$'s assigned.

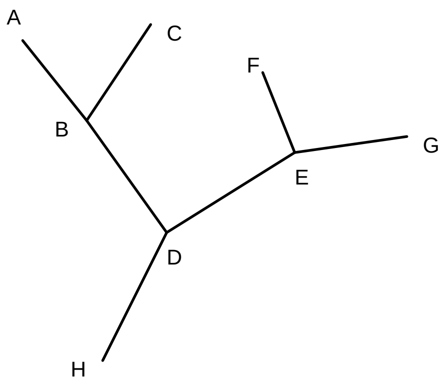

Fig.3

For a given value h_B of the B water level it's defined: with (17), diffluent discharge in border sides having end B; based on continuity, discharge Q in side BD; based on (17), water level h_D.

In internal node E, value h_E depends, then, both on conditions imposed in border nodes F and G, and by water level h_D. Such conditions allow for defining water level h_E and discharge in side ED which respects nodal continuity.

This observation's the starting point of a calculus algorithm for verifying branch networks.

Let's again consider the schematic situation in fig.3.

The numeric procedure starts assigning, to water level in B, a first position value $h_B^{(1)}$.

Having introduced such a value in (17), discharge in sides AB and CB are calculated and, for continuity, discharge in BD. This latter put in (17) determines h_D and, even if slightly dealt with above, water level in E and discharge in sides FE, GE, ED.

Continuity condition in node D permits the calculation of both diffluent discharge in side DH, and water level $h_H^{(1)}$, to compare with value $\bar{h}_H$ imposed in the same node.

Indicated thru $\lambda = \left|h_H^{(1)} - \bar{h}_H\right|$ the absolute assumed value from the difference between the calculated water level and that given in H, if it results in 1 being greater than a prefixed approximation, the procedure's repeated starting from a new value $h_B^{(2)}$ automatically taken through a dichotomy method.

4. APPLICATION IN A CHANNEL NETWORK

The procedure shown has been applied in resolving the verifying problem in channel network shown in Fig. 4.

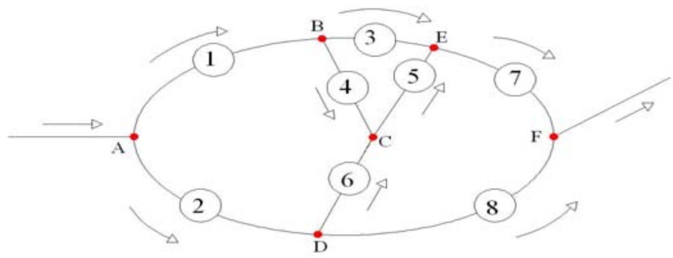

Fig.4

The network's formed by 8 trapezoidal channels (1.5H:1.0V), with α=1, of whose characteristics are reported in tab.1. In all channels flow's subcritical, arrows indicating flow direction. Extreme conditions in valley node (node F) are given by h_F=5.0 m and Q=250m^3/s.

channel	length(m)	width(m)	reach(m)	n=1/k	k	i_f
1	200	30	50	0.0130	76.92	0.005
2	200	40	50	0.0130	76.92	0.005
3	200	20	50	0.0120	83.73	0.005
4	100	20	25	0.0140	71.43	0.005
5	100	20	25	0.0130	76.92	0.005
6	100	25	25	0.0130	76.93	0.005
7	100	30	25	0.0140	71.43	0.005
8	300	50	75	0.0140	71.43	0.005

Table 1. Input data for channel network example

In table 2 border nodes water height values assigned are reported and for internal nodes, h_{min} and h_{max} values deduced according to the above obtained.

Node	h (m)	h_{min} (m)	h_{max} (m)
A	-	0.6	9.9
B	-	1.8	8.1
C	-	2.7	6.95
D	-	1.8	6.35
E	-	3.8	5.9
F	5.00	-	-

Table 2

In table 3, water level values relative to h internal nodes of each channel, water level values in internal points in each channel, and the various channel discharge distribution obtained in the described procedure, and the values shown by [5] are reported.

We have indicated in h* and Q* with [5] calculated values, and h and Q values calculated with the described algorithm .

	section	distance(m)	h(m)	h*(m)
Channel 1	1	0	2.4567	2.4553
Q =104.8056 m^3/s	2	50	2.7184	2.7176
Q^*=104.9785 m^3/s	3	100	2.9771	2.9767
	4	150	3.2335	3.2336
	5	200	3.4886	3.4889
Channel 2	1	0	2.4529	2.4405
Q =145.1931 m^3/s	2	50	2.7164	2.7047
Q^*=145.0214 m^3/s	3	100	2.9763	2.9651
	4	150	3.2337	3.2230
	5	200	3.4896	3.4791
Channel 3	1	0	3.4886	3.4982
Q =59.0052 m^3/s	2	50	3.7412	3.7510
Q^*=59.6867 m^3/s	3	100	3.9933	4.0033
	4	150	4.2450	4.2552
	5	200	4.4964	4.5067
Channel 4	1	0	3.4913	3.5101
Q =45.7899 m^3/s	2	25	3.6170	3.6359
Q^*=45.2919 m^3/s	3	50	3.7486	3.7616
	4	75	3.8682	3.8872
	5	100	3.9937	4.0128
Channel 5	1	0	3.9941	4.0077
Q =51,7511 m^3/s	2	25	4.1198	4.1334
Q^*=51.0382 m^3/s	3	50	4.2454	4.2590
	4	75	4.3711	4.3846
	5	100	4.4964	4.5101
Channel 6	1	0	3.4937	3.5222
Q =5.9498 m^3/s	2	25	3.6187	3.6472
Q^*=5.7463 m^3/s	3	50	3.7437	3.7722
	4	75	3.8687	3.8973
	5	100	3.9937	4.0223
Channel 7	1	0	4.4964	4.4916
Q =110.7571 m^3/s	2	25	4.6224	4.6177
Q^*=110.7249m^3/s	3	50	4.7483	4.7437
	4	75	4.8742	4.8696
	5	100	5.00	4..9954
Channel 8	1	0	3.4897	3.4905
Q =139.2432 m^3/s	2	75	3.8684	3.8694
Q^*=139.2751 m^3/s	3	150	4.2462	4.2473
	4	225	4.6233	4.6246
	5	300	5.00	5.0013

Table 3. Discharge and flow depths for channel network example

The comparison of the solutions obtained, allows assertation that the algorithm described, in sub critical flow, correctly responds and then can be considered a valid tool for such a study, showing in conclusion that gradually varied flows in open channel networks can be analysed directly at the computer.

In another note, results obtained through experiments will be reported.

ACKNOWLEDGMENTS

The Author wish to tank Prof. Aniello Russo Spena for the generous and useful suggestions.

REFERENCES

[1] A. Russo Spena, M. T. Todisco and G. Vacca, *Profili di corrente in reti di canali apert*i, L'Acqua, Italia, vol.3, pp 33-39, Maggio-Giugno,1996.

[2] G. Evangelisti, Sui problemi di rigurgito in alvei cilindrici di sezione generica, dal Volume "Giuseppe Evangelisti:Scritti scelti", Bologna, Italia, Settembre, 1982.

[3] G. Evangelisti, Mathematical Models of Open-Channel Flow, Proceedings of the NATO Advanced Study Institute on "Surface Hydrodynamics", Bressanone, Aug.29-Sept 7, 1966

[4] Andrea Russo Spena, Sul tracciamento dei profili di corrente in alvei cilindrici, L'Energia Elettrica, vol XXVII, n.7,1950..

[5] A.M. Schulte and M. H. Chaudry, Gradually-varied Flow in Open Channel Networks, Journal of Hydraulic Research, vol. 25, nº. 3, 1987.