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PERFORMANCE COMPARISON OF THREE ALGORITHMS FOR TWO-CHANNEL SINEWAVE PARAMETER ESTIMATION: SEVEN PARAMETER SINE FIT, ELLIPSE FIT, SPECTRAL SINC FIT

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Abstract − The comparison of three different algorithms for the estimation of parameters of two sine signals with common frequency is presented. The algorithms are the ellipse fit, the seven parameter sine fit and the spectral sinc fit. The comparison includes signal to noise ratio analysis, amplitude analysis and phase difference analysis.

Keywords: sinewave parameter estimation, amplitude and phase measurements.

1. INTRODUCTION

Estimation of sinewave parameters is needed in many applications of instrumentation and measurement. Driven by the need to standardize algorithms for analog to digital converters, IEEE included in the 1057 standard [1] two algorithms that are simple to implement, very efficient and accurate. The first algorithm estimates the amplitude, phase and DC component for a known signal frequency. It is called the three-parameter sine-fit and it is a multiple linear regression that requires no initial estimatives of the parameters and the optimal parameters are obtained without iterations. However, their accuracy depends heavily on the accuracy at which the signal frequency is known. To address this issue, the second algorithm also estimates the signal frequency (or more accurately, it estimates the ratio between the signal frequency and the sampling rate). Unfortunately, it becomes a nonlinear regression with the addition of the frequency to the list of estimated parameters. The solution presented in [1] requires an initial set of estimatives of the four parameters which are then improved with each iteration.

In some applications, it is necessary to estimate the parameters of two common frequency sine signals (*e.g.*, impedance measurements [2], laser anemometry [3] and linear system single tone characterization). In these cases, the methods presented in [1] are not suitable because they don't take into account the system restriction that both signals have the same frequency.

In this paper, the comparison between three algorithms that are capable of estimating the parameters of two common frequency sine signals is presented.

2. THE ALGORITHMS

This section describes the three algorithms compared in this paper: ellipse fit; seven parameter sine fit; and spectral sinc fit. The goal of these algorithms is to estimate the amplitudes D_i, phases ϕ_i, DC components C_i and common frequency f of two acquired sinewaves modeled by

$$u_i(t) = D_i \cos(2\pi f t + \phi_i) + C_i \,. \tag{1}$$

In most two-channel applications the value of the amplitude and phase of each channel is not required. The only values needed are in fact the amplitude ratio D_2/D_1 and phase difference $\Delta\varphi = \phi_2 - \phi_1$. However, the frequency f must also be estimated since it is not accurately known. This is due to the uncertainty of the generated sinewaves frequency and uncertainty of the sampling frequency f_s.

The Cramér-Rao lower bound (CRB) for parameter estimation of dual-channel sinewaves was determined in [4].

2.1. Ellipse Fit

The ellipse fit algorithm was originally developed by [5] and improved by [6]. It relies on estimating the parameters of the ellipse that best fit, in a least-squares sense, the X-Y plot of two sinewaves. The ellipse is mathematically described by the conic

$$F(u_1,u_2) = a u_1^2 + b u_1 u_2 + c u_2^2 + d u_1 + e u_2 + g = 0 \tag{2}$$

with the constraint $b^2 - 4ac < 0$ which can be transformed into $b^2 - 4ac = -1$ by scaling. The algorithm consists on a non-iterative constrained minimization process based of Lagrange multipliers [6].

The amplitude ratio and phase difference are determined from the ellipse parameters [7]

$$\frac{D_2}{D_1} = \sqrt{\frac{a}{c}} \,, \quad \cos(\Delta\varphi) = -\frac{\operatorname{sign}(a)\, b}{2\sqrt{ac}} \,. \tag{3}$$

The sign of the phase difference is obtained by observing the rotation direction of the consecutive samples. To avoid miscalculations due to the presence of noise, a voting system was implemented to obtain the rotation direction [7].

The numerical implementation of this algorithm requires the construction of 3×3 matrices with a total of only 15 different elements. This is a major advantage in terms of memory requirements making the algorithm suitable for DSP implementation, since the amount of memory needed is independent of the number of acquired samples N.

2.2. Seven parameter sine fit

The seven parameter sine fit algorithm [8] is a two channel extension of the four parameter sine fit algorithm normalized in [1] for the characterization of ADCs. The samples of the two acquired sinewaves are used simultaneously to estimate the amplitudes D_i, phases ϕ_i, DC components C_i and the common frequency f. The need to estimate the frequency makes the algorithm nonlinear and iterative. From the initial estimates of the sinewaves parameters a system of nonlinear equations yields new estimatives and a frequency correction Δf to be used in the next iteration. The convergence of the algorithm is dependent on the number of samples and the initial estimates used. This algorithm involves the creation of a matrix of size $2N\times7$. As the number of samples increases, the memory requirements will limit the algorithm applicability in DSP implementation.

2.3. Spectral sinc fit

The spectral sinc fit algorithm has been recently proposed as a new method to estimate the parameters of an acquired sinewave [9]. This method has been extended to be applied to two channel acquisitions. The acquisition of a limited number of samples is equivalent to applying a rectangular window to the sinewaves. The theoretical spectrum of a windowed sinewave is

$$\hat{X}_i(\omega) = \frac{D_i}{2}\left[W(\omega-\omega_X)\,e^{j\varphi_i} + W(\omega+\omega_X)\,e^{-j\varphi_i}\right] \tag{4}$$

where $W(\omega)$ is the spectrum of a rectangular window (*i.e.*, a sinc function). The resulting two-sided spectrum $\hat{X}_i(\omega)$ consists on two overlapping sinc functions centered at $\pm\omega_X = \pm 2\pi f/f_s$. The maximums of $\hat{X}_i(\omega)$ are not centered at the frequencies $\pm\omega_X$ due to the leakage of one sinc into the other.

The algorithm searches for the sinewaves parameters that minimize the cost function

$$\varepsilon = \sqrt{\sum_{\omega}\left|\hat{X}_1(\omega) - X_1(\omega)\right|^2 + \left|\hat{X}_2(\omega) - X_2(\omega)\right|^2} \tag{5}$$

where $X_i(\omega)$ is the spectrum of each acquired channel. The algorithm is iterative and uses as a first estimate the frequency obtained by the IpDFT [10]. The remaining initial parameters are obtained by applying the three parameter sine fit to each channel. The main advantage of this algorithm is that the iterative part can be accurately computed using as little as three sample points per channel, making it memory wise very efficient (only the initial FFTs are done with the full number of acquired samples).

3. NUMERICAL SIMULATIONS

To assess the performance of the different algorithms, they were implemented in Matlab and several tests were executed. Since the ultimate goal is to estimate the amplitude ratio (D_2/D_1) and the phase difference ($\Delta\varphi$), the tests estimated the amplitude ratio error (*i.e.*, the difference between the estimated amplitude ratio and the imposed ratio) as well as the phase difference error. For each set of tested parameters, 10 000 different runs were executed to obtain the average values and the corresponding standard deviations. In each run, the initial phase of the first channel (ϕ_1) is a random variable with a uniform pdf between -180º and 180º. Signal frequency is 1 kHz and 1920 samples per channel are taken at 96 kS/s. White Gaussian noise is added according to each channel signal-to-noise-ratio (SNR).

3.1. Signal to noise ratio analysis

In this analysis, the signal amplitudes are fixed at D_2=1 V and D_1=0.5 V. Since it is known that the ellipse fit algorithm cannot work near $\Delta\varphi$=180° and $\Delta\varphi$=0° because of ellipse degeneration, the phase difference is a uniform pdf in the $\pm$[10º;170º] range. This issue will be analyzed and discussed in Section 3.3.

The results for the ellipse fit are shown in Fig. 1 and Fig. 2. It can be seen that the algorithm is biased for poor signal to noise ratios (typically below 40 dB). As expected, the standard deviations improve with the increase in SNR.

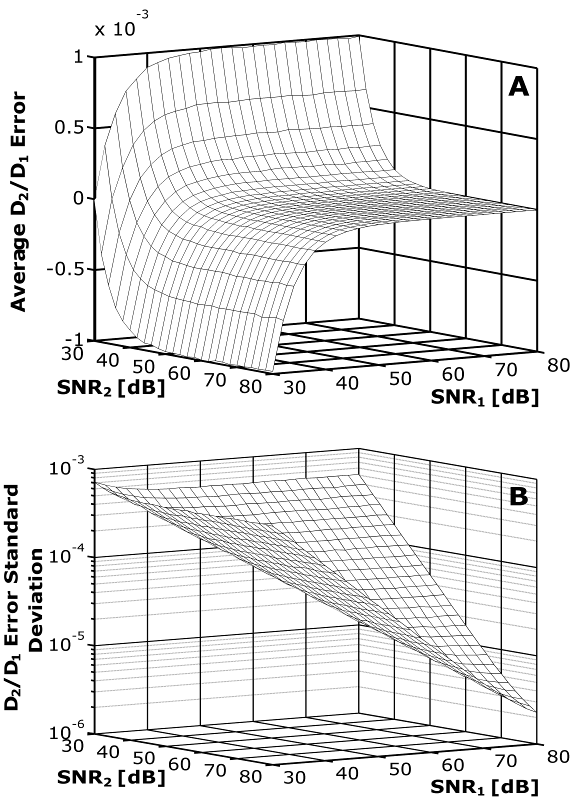

Fig. 1. Average amplitude ratio error (A) and corresponding standard deviation (B) for the ellipse fit algorithm as a function of the two signal to noise ratios for D_1=1 V and D_2=0.5 V.

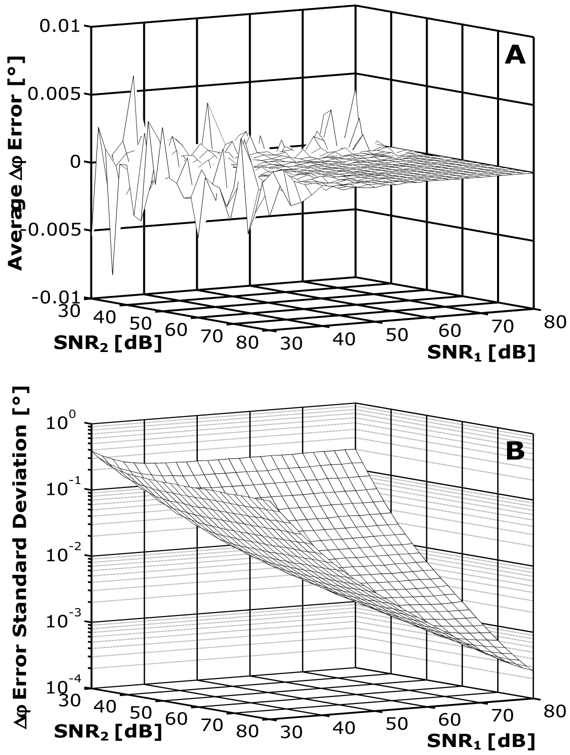

Fig. 2. Average phase difference error (A) and corresponding standard deviation (B) for the ellipse fit algorithm as a function of the two signal to noise ratios for D_1=1 V and D_2=0.5 V.

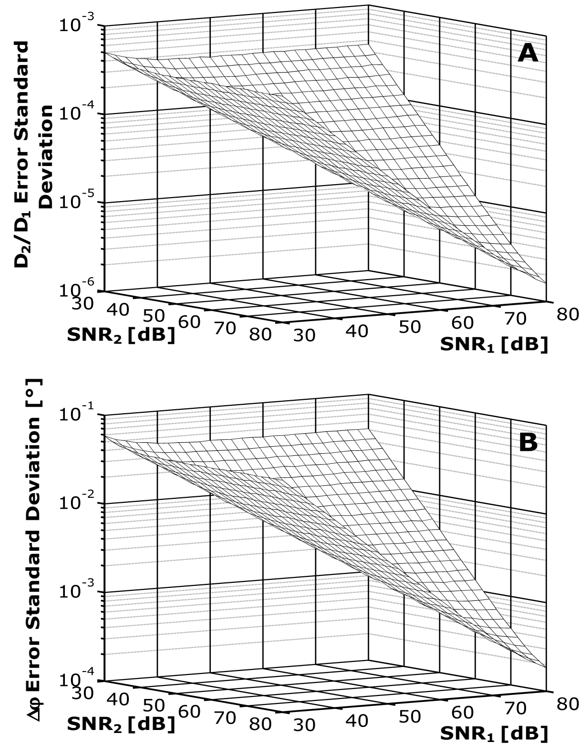

Fig. 3. Standard deviation of the amplitude ratio (A) and phase difference error (B) for the seven parameter sine fit algorithm as a function of the two signal to noise ratios for D_1=1 V and D_2=0.5 V.

Note that, the fluctuations in the results of the average phase errors (Fig. 2 A) for the worst signal to noise ratios are caused by the low number of repetitions and that the corresponding standard deviations are considerably higher than the represented fluctuation (e.g., for SNR=30dB the average error is -0.005° and the standard deviation is 0.4°).

The results for the seven parameter sine fit are shown in Fig. 3 while the results for the spectral sinc fit are presented in Fig. 4. These algorithms are not biased and so the results that are shown correspond only to the standard deviations. Note that, the evolutions of the standard deviations are quite similar for these algorithms. Comparing with the ellipse fit algorithm, the evolution pattern is the same, but the standard deviation of the phase error is higher for the ellipse fit.

3.2. Amplitude analysis

In this section, the amplitude analysis of the three algorithms is presented. The signal to noise ratios are set to 60 dB and the amplitudes are swept from 0.1 V up to 2 V with 0.1 V resolution. For these situations, the phase difference error standard deviation is independent on the signal amplitudes (*i.e.*, for this analysis, it is constant and the values are presented in Table 1 for different SNR values). For these SNR values, the ellipse fit is not biased as shown in Fig. 1 and Fig. 2.

The results in Fig. 5 represent the amplitude ratio error relative standard deviation. It can be seen that the lowest values are obtained for higher value of D_1. This is caused by the fact that the amplitude of the first channel is the denominator of the amplitude ratio and D_2 is the numerator.

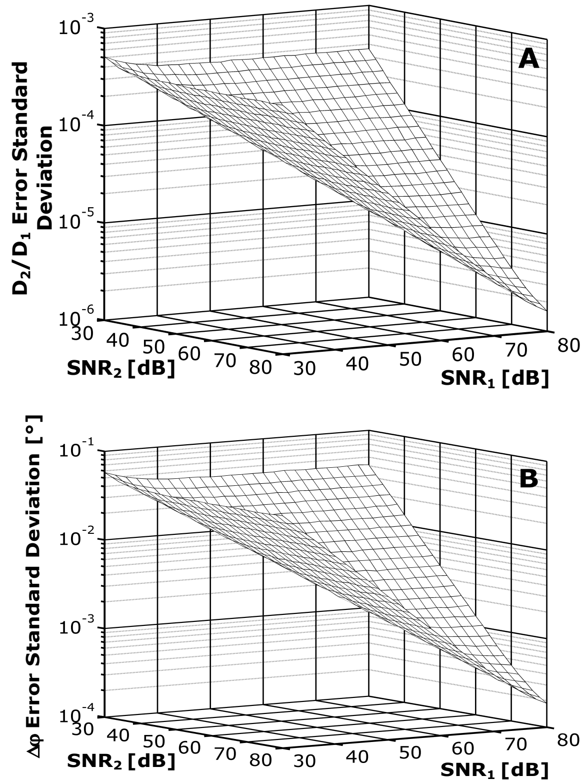

Fig. 4. Standard deviation of the amplitude ratio (A) and phase difference error (B) for the spectral sinc fit algorithm as a function of the two signal to noise ratios for D_1=1 V and D_2=0.5 V.

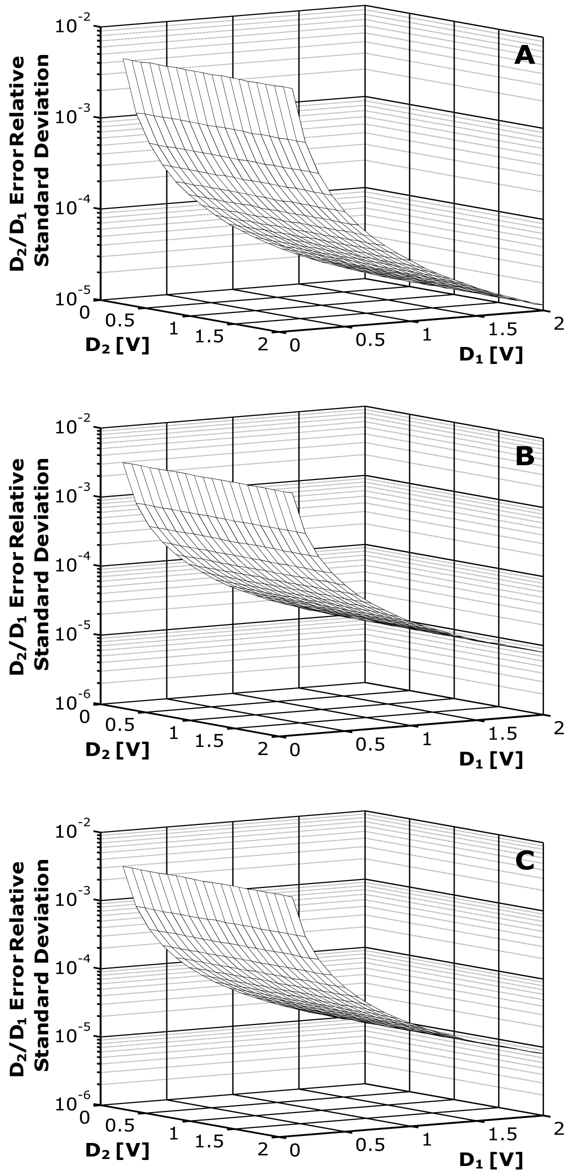

Fig. 5. Relative standard deviation of the amplitude ratio for the ellipse fit (A), seven parameter sine fit (B) and spectral sinc fit (C) as a function of the two signal amplitudes for SNR_1=SNR_2=60 dB.

3.3. Phase analysis

Regarding the phase analysis, the tests that were performed used D_2=1 V, D_1=0.5 V and three different values of the common SNR. The imposed phase difference was swept from -180º up to 180º with resolution 0.005º. As expected, the seven parameter sine fit and the spectral sinc fit algorithms are independent on the phase difference (results presented in Table 1).

The ellipse fit algorithm is quite different. Due to ellipse degeneration, the algorithm has problems for phase differences near 0º and 180º (as shown in Fig. 6 and with more detail in Fig. 7). The range of affected phase difference values depends on the SNR values. Remarkably, in spite of the ellipse degeneration, the algorithm is capable of estimating the amplitude ratio without bias and with the same standard deviation for all phase difference values.

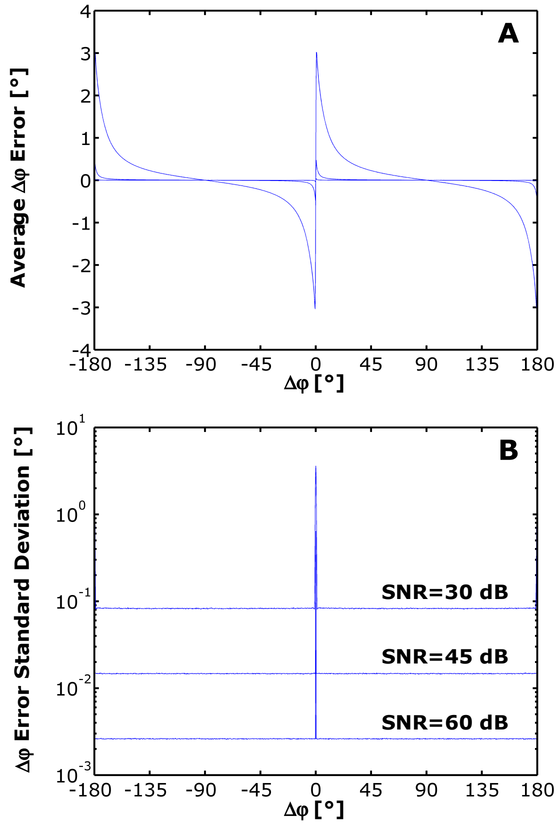

Fig. 6. Average phase difference error (A) and standard deviation (B) for the ellipse fit algorithm as a function of the phase difference and common SNR for D_1=1 V and D_2=0.5 V.

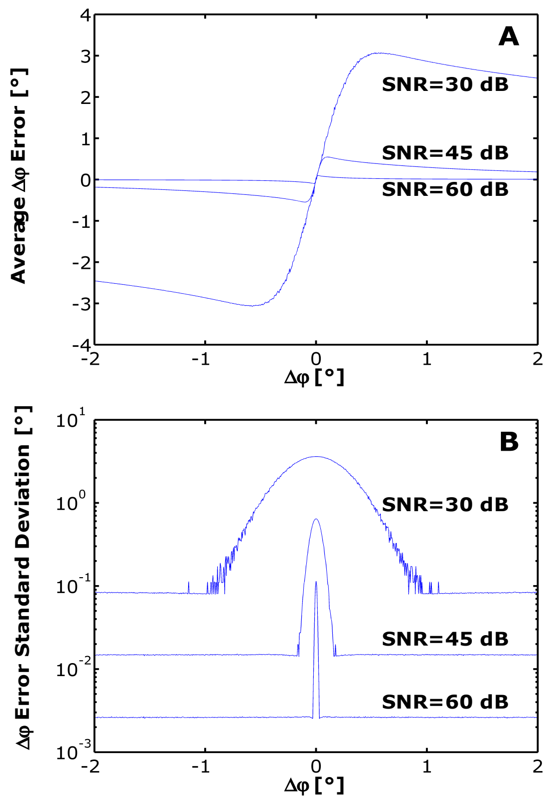

Fig. 7. Detailed view of Fig. 6 near Δφ=0º.

The results presented in Table 1, correspond to the values obtained with the three different algorithms for different values of signal to noise ratios obtained for the phase difference of 90º (to avoid problems with the ellipse fit algorithm), D_1=1 V and D_2=0.5 V.

Also included in Table 1 for comparison are the Cramér-Rao bounds determined using the equations derived in [4]. For the relative amplitude ratio, the standard deviation that corresponds to the bound is

$$\frac{\sigma_{D_2/D_1}}{D_2/D_1}=\sqrt{\frac{\mathrm{SNR}_1+\mathrm{SNR}_2}{N\,\mathrm{SNR}_1\,\mathrm{SNR}_2}} \tag{6}$$

while the standard deviation that corresponds to the bound of the phase difference is

$$\sigma_{\Delta\varphi}\left[^\circ\right]=\frac{180}{\pi}\sqrt{\frac{\mathrm{SNR}_1+\mathrm{SNR}_2}{N\,\mathrm{SNR}_1\,\mathrm{SNR}_2}} \tag{7}$$

where

$$\mathrm{SNR}_i=\frac{D_i^2}{2\sigma_i^2} \tag{8}$$

and σ_i^2 is the variance of the zero-mean Gaussian white noise of channel i.

Note that the results from the seven parameter sine fit are identical to the ones obtained with the spectral sinc fit and also identical to the Cramér-Rao bound. The results of the ellipse fit are slightly worse.

Table 1. Comparison of the algorithms for Δφ=90º.

		SNR₁=SNR₂		
		30 dB	45 dB	60 dB
Ellipse fit	D_2/D_1 relative standard deviation	1.5×10^{-3}	2.6×10^{-4}	4.6×10^{-5}
	Phase difference standard deviation [°]	8.3×10^{-2}	1.5×10^{-2}	2.6×10^{-3}
Seven parameter sine fit	D_2/D_1 relative standard deviation	1.0×10^{-3}	1.8×10^{-4}	3.2×10^{-5}
	Phase difference standard deviation [°]	5.8×10^{-2}	1.0×10^{-2}	1.8×10^{-3}
Spectral sinc fit	D_2/D_1 relative standard deviation	1.0×10^{-3}	1.8×10^{-4}	3.2×10^{-5}
	Phase difference standard deviation [°]	5.9×10^{-2}	1.0×10^{-2}	1.8×10^{-3}
Cramér-Rao bound	D_2/D_1 relative standard deviation	1.0×10^{-3}	1.8×10^{-4}	3.2×10^{-5}
	Phase difference standard deviation [°]	5.8×10^{-2}	1.0×10^{-2}	1.8×10^{-3}

4. CONCLUSIONS

The performance of three algorithms for two-channel sinewave parameter estimation was analyzed. The ellipse fit method is biased for low SNRs and has a slightly lower precision in the phase difference estimation. The amplitude analysis shows that the three algorithms perform almost identically. The seven parameter sine fit and spectral sinc fit algorithms are phase independent, while the ellipse fit algorithm suffers from the ellipse degeneration phenomena. Despite this, the ellipse fit estimates the amplitude ratio without bias and with similar precision as the other algorithms for SNR>40 dB.

For equal signal to noise ratios, the standard deviations of the estimated parameters are identical to the CRB for the seven parameter sine fit and spectral sinc fit.

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