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STATISTICAL ANALYSIS OF THE WORD ERROR RATE MEASUREMENT IN ANALOG-TO-DIGITAL CONVERTERS

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Abstract – The word error rate (WER) in an Analog to Digital Converter (ADC) is the probability to receive a wrong code for an input, after correction is made for gain, offset, and nonlinearity errors, and a specified allowance is made for noise. Typical causes of word errors are metastability and timing jitter of comparators within the ADC [1].

The proposed paper is an evolution of previous work [6] and focuses the attention to the word error rate estimation and to the Annex A of IEEE standard 1241. Other statistical techniques, which can better integrate what is sustained in the standard and in [2], have been proposed. In particular, Chi-square and F distributions have been introduced for a more accurate measurement of the word error rate in the case of n successive observations.

Keywords: Analog to Digital Converter (ADC), IEEE Std 1241, Word Error Rate (WER).

1. SOME THEORETICAL CROSS-REFERENCES

As we have seen in [6] the sum of the square of ν normal variables in the reduced form and reciprocally independent, is a chi-square variable which can assume any value between zero and infinite with probability density function:

$$f_\nu(m) = \frac{m^{\frac{\nu}{2}-1}}{2^{\frac{\nu}{2}}\Gamma\left(\frac{\nu}{2}\right)} e^{-\left(\frac{m}{2}\right)} \; ; \qquad 0 \le m < +\infty \tag{1}$$

with $\Gamma(\alpha)$ it is still defined in [6]. For $\nu=1$, we have $\chi_1^2 = N_1^2(0,1)$. From eq. (1) it is possible to deduce that this distribution is asymmetric with mode equal to $(\nu-2)$, expected value ν and variance 2ν. Incidentally we notice that the positive square root of a chi-square χ_ν^2, that is $\chi_\nu = \sqrt{\sum_{i=1}^{\nu} N^2(0,1)}$, presents a probability density function which, for ($\nu=2$), relapsing in the *Rayleigh distribution* with expected value $E\{\chi_2^2\} = E\left\{\sqrt{N_1^2(0,1) + N_2^2(0,1)}\right\} = \sqrt{\frac{\pi}{2}}$, while for ($\nu=3$) it relapses into that of *Maxwell* with the expected value $E\{\chi_3^2\} = E\left\{\sqrt{N_1^2(0,1) + N_2^2(0,1) + N_3^2(0,1)}\right\} = 3 - \frac{4}{\pi}$.

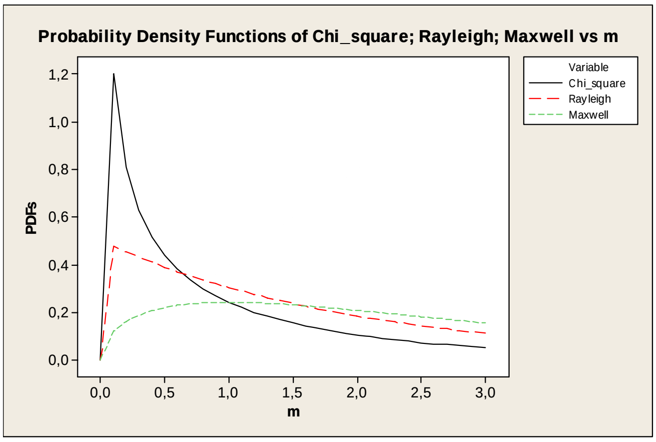

Figure 1. Chi square, Rayleigh and Maxwell Probability Density Functions.

To finish this brief theoretical introduction it's possible to demonstrate, remembering how this distribution was introduced, that the sum of two independent chi-square, respectively with ν_1 and ν_2 degrees of freedom, is still a chi-square with ($\nu_1+\nu_2$) degrees of freedom:

$$\chi_{\nu_1+\nu_2}^2 = \chi_{\nu 1}^2 + \chi_{\nu_2}^2 \tag{2}$$

and it is to be remembered that this peculiarity can be extrapolated, repeatedly, not only to any number of independent chi-square but also considering the difference that is for ($\nu_2 > \nu_1$), we obtain:

$$\chi^2_{\nu_2-\nu_1} = \chi^2_{\nu_2} - \chi^2_{\nu_1} \qquad (3)$$

2. APPLICATION OF THE CHI-SQUARE DISTRIBUTION IN THE ESTIMATION OF THE WORD ERROR RATE

Taking into account n independent successive observations $(o_1, o_2, \cdots, o_n)$ and assuming [6] each observation as a normally distributed random variable with expected value m_o and standard uncertainty u_o, the *chi-square* distribution with *(n-1)* degrees of freedom can be represented as $\frac{\sum_{i=1}^{n}(o_i - \bar{o})^2}{u_0^2} = \chi^2_{n-1}$ where the mean $\bar{o} = \frac{\sum_{i=1}^{n} o_i}{n}$ of such variables is also normally distributed with the mean value m_o and reduced variance equal to $\frac{u_0^2}{n}$.

The uncertainty interval can be introduced by considering the *chi-square* probability distribution with ν degrees of freedom is associated. With the pre-arranged confidence level p, the interval is defined as:

$$P\left\{\chi^2_\alpha \le \chi^2_\nu \le \chi^2_{p+\alpha}\right\} = F_\nu\left(\chi^2_{p+\alpha}\right) - F_\nu\left(\chi^2_\alpha\right) = p \qquad (4)$$

where α is a value in the range from zero to *(1-p)*, while the extremes of the interval χ^2_α e $\chi^2_{p+\alpha}$ are respectively α- and $(p+\alpha)$-quantiles of the distribution function of χ^2_ν defined from:

$$F_\nu(m) = P\left\{\chi^2_\nu \le m\right\} = \int_0^m f_\nu(m')dm' \qquad (5)$$

being *f(m')* is obtained from Eq.(1). As is already known, a β-quantile is an m_β value so that $F_\nu(m_\beta) = \beta$. Such quantiles are tabulated in different values of degrees of freedom ν corresponding to the respective β but can be obtained more efficiently from statistic software such as, for instance, Minitab©.

3. APPLICATION OF THE "F" DISTRIBUTION IN THE ESTIMATION OF THE WORD ERROR RATE

The ratio of two independent chi-square variables, each divided by its respective degrees of freedom, is a random variable F which follows the distribution:

$$F(\nu_1, \nu_2) = \frac{\chi^2_{\nu_1}/\nu_1}{\chi^2_{\nu_2}/\nu_2} \; ; \qquad 0 \le F < +\infty \qquad (6)$$

The probability density function of F is given by the following expressions:

$$f(m; \nu_1, \nu_2) = \frac{\Gamma[(\nu_1+\nu_2)/2]}{\Gamma(\nu_1/2)\Gamma(\nu_2/2)} \nu_1^{\nu_1/2} \nu_2^{\nu_2/2} \frac{m^{(\nu_1/2)-1}}{(\nu_1 m + \nu_2)^{(\nu_1+\nu_2)/2}} \qquad (7)$$

where $\Gamma(\alpha)$ is still defined in [6]. The distribution is asymmetric and in this case the β-quantiles $m_\beta(\nu_1, \nu_2)$ are defined as:

$$P\left\{F(\nu_1,\nu_2) \le m_\beta(\nu_1,\nu_2)\right\} = \int_0^{m_\beta(\nu_1,\nu_2)} f\;(m, \nu_1, \nu_2)\, dm = \beta \qquad (8)$$

This can be verified by using (6) and (7):

$$m_{(1-\beta)}(\nu_2, \nu_1) = \frac{1}{m_\beta(\nu_1, \nu_2)} \qquad (9)$$

As an application of the F distribution in this paper we take into consideration an application the example presented in the GUM [5] par. H5.

Let's consider a set of n repeated observations throughout each day and we suppose that such a set is reproduced in following days. We denote as v_{jk} the random variable associated with k-observation throughout the j-day.

The model adopted can be represented as:

$$\begin{cases} V_{jk} = m_0 + G_{jk} + T_j; & j = 1, \cdots; m;\; k = 1, \cdots; n; \\ \overline{V_{j.}} = \sum_{k=1}^{n} V_{jk}/n = m_0 + \overline{G_{j.}} + T_j; & \text{with } \overline{G_{j.}} = \sum_{k=1}^{n} G_{jk}/n \\ \overline{\overline{V..}} = \sum_{j=1}^{m} \overline{V_j}/m = m_0 + \overline{\overline{G.}} + \overline{T.}; & \text{with } \overline{\overline{G.}} = \sum_{j=1}^{m} \overline{G_{j.}}/m;\; \overline{T.} = \sum_{j=1}^{m} T_j/m \end{cases} \qquad (10)$$

G_{jk} and T_j denote the random errors, with expected values zero, which distinguish respectively the variability within day (assumed from day to day – *within variability*) and the variability between days (in periods of time such as, for example, weeks, months, years – *between variability*). We hypothesize that the distribution of the model is normal, so that:

$$G_{jk} = N\left(0,\sigma_G^2\right);\ \overline{G}_{j.} = N\left(0,\sigma_G^2/n\right);\ T_J = N\left(0,\sigma_T^2\right);$$
$$\overline{G}_j + T_J = N\left(0,\sqrt{\sigma_G^2/n+\sigma_T^2}\right);$$
$$\overline{\overline{G}} = N\left(0,\sigma_G/\sqrt{mn}\right);\ \overline{T} = N\left(0,\sigma_T/\sqrt{m}\right); \tag{11}$$
$$\overline{\overline{G}} + \overline{T} = N\left(0,\sqrt{\sigma_G^2/n+\sigma_T^2}/\sqrt{m}\right)$$

Due to the hypothesized independence among the observations, the random errors of the model are independent both in the field of the $\{G_{jk}\}$, in that of $\{T_j\}$ and in that of mutual behavior.

Consequently we have the following property:

$$\begin{cases} \left(V_{jk} - \overline{V_j}\right) \text{ is independent from } \overline{V_j} \\ \text{and therefore from } \overline{G_j} \\ \left(\overline{V_j} - \overline{\overline{V}}\right) \text{ is independent from } \overline{\overline{V}} \\ \text{and therefore from } \overline{\overline{G}} + \overline{T} \end{cases} \tag{12}$$

At this point let's consider the following equation:

$$\sum_{k=1}^{n} \frac{\left(V_{jk} - \overline{V_j}\right)^2}{\sigma_G^2} = \sum_{k=1}^{n} \frac{G_{jk}^2}{\sigma_G^2} - \frac{\overline{G}_j}{\sigma_G^2/n} = \chi_{j,n}^2 - \chi_{j,1}^2 = \chi_{j,n-1}^2 \tag{13}$$

Whereas with $\chi_{j,\nu}^2$ the chi-square associated with the *j* day with ν degrees of freedom where we take into account the relation (3) and the second equation of (12).

Summing up eq. (13) with respect to *j*, bearing for eq. (2) that $\sum_{j=1}^{m} \chi_{j,n-1}^2 = \chi_{m(n-1)}^2$ due to the independence from day to day and dividing by the degrees of freedom *m(n-1)*, we can suppose:

$$\frac{\sum_{j=1}^{m} \sum_{k=1}^{n} \left(V_{jk} - \overline{V_j}\right)^2}{m\,(n-1)} = \sigma_G^2 \frac{\chi_{m(n-1)}^2}{m(n-1)} \tag{14}$$

where the first member is an unbiased estimator $\tilde{\sigma}_G^2$ of σ_G^2 being its expected value the same as σ_G^2 (taking into account that $E\{\chi_\nu^2\} = \nu$).

Let's now consider another important quantity, that is:

$$\frac{\sum_{j=1}^{m} \left(\overline{V}_j - \overline{\overline{V}}\right)^2}{\sigma_G^2/n+\sigma_T^2} = \frac{\sum_{j=1}^{m} \left(\overline{G}_j + T_j\right)^2}{\sigma_G^2/n+\sigma_T^2} - \frac{\left(\overline{\overline{G}} + \overline{T}\right)^2}{\left[\sigma_G^2/n+\sigma_T^2\right]/m} = \tag{15}$$
$$= \chi_m^2 - \chi_1^2 = \chi_{m-1}^2$$

where the last parameter is, as usual, accordable to the (3) and the second property (12) has been taken into account.

By analogy with eq. (14) we can introduce:

$$\frac{\sum_{j=1}^{m} \left(\overline{V}_j - \overline{\overline{V}}\right)^2}{m-1} = \left[\frac{\sigma_G^2}{n} + \sigma_T^2\right] \frac{\chi_{m-1}^2}{m-1} \tag{16}$$

affirming that the first member represents an unbiased estimator of $\left[\frac{\sigma_G^2}{n} + \sigma_T^2\right]$.

From the two estimators of eqs. (14) and (16) we can also introduce an unbiased estimator of σ_T^2 , as:

$$\tilde{\sigma}_T^2 = \frac{\sum_{j=1}^{m} \left(\overline{V}_j - \overline{\overline{V}}\right)^2}{m-1} - \frac{\sum_{j=1}^{m} \sum_{k=1}^{n} \left(V_{jk} - \overline{V}_j\right)^2}{mn(n-1)} \tag{17}$$

Finally, working on the ratio between eq. (14) and eq. (16) and remembering the definition of (6), we have:

$$F\left(m[n-1],m-1\right) = \frac{\frac{\sigma_G^2}{n} + \sigma_T^2}{\sigma_G^2} \frac{\tilde{\sigma}_G^2}{\frac{\tilde{\sigma}_G^2}{n} + \tilde{\sigma}_T^2} \tag{18}$$

In the particular case where the contribution of between-group variability (from day to day) is null, therefore $\sigma_T^2 = 0$, eq. (18) can be simplified as:

$$F\left(m[n-1],m-1\right) = \frac{\tilde{\sigma}_G^2}{\tilde{\sigma}_G^2 + n\tilde{\sigma}_T^2} \tag{19}$$

Equation (19) can also be useful to test the hypothesis of the insignificance of the variability from day to day ($\sigma_T^2 = 0$). For this purpose, in eq. (19), the estimators are substituted by the corresponding values (called "estimation") obtained in the specific measurement. If the value of F(*,*) so obtained is superior to the *0,95*-quantile of the F distribution, (for example) it's permitted to reject the hypothesis and therefore maintain that the variability from day to day is statistically significant with a risk of 5%.

4. CONCLUSIONS

The aim of this paper is to introduce other detailed methods as a contribution to define the Word Error Rate measurement and the Annex A of the IEEE Std 1241 [1]. This is made according with GUM [5] and its supplements carried out by BIPM. It has been quantified the uncertainty level with relative confidence level in the two cases of *Chi-*

square and *F* distributions, with the final aim to give a contribution for a new draft of such standard.

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