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SIGNIFICANCE OF CORRELATION IN THE UNCERTAINTY EVALUATION OF SAMPLING OSCILLOSCOPE MEASUREMENTS

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Abstract − We show that correlations in the estimate of the impulse response of ultrafast sampling oscilloscopes might play an important role for uncertainty evaluations. This is demonstrated by determining the uncertainty associated with the estimate of the oscilloscope's input signal which is calculated using the output signal of the oscilloscope and the impulse response. We observe that the resulting uncertainty depends on the size of the correlation in the impulse response and we conclude that such correlations should be accounted for in an uncertainty analysis.

Keywords: sampling oscilloscope, dynamic uncertainty, Monte-Carlo method

1. INTRODUCTION

Ultrafast sampling oscilloscopes are routinely used for precise measurements of microwave waveforms and the characterization of high-speed electronics. Such instruments are an ideal tool for this purpose since they are portable, easy to use and relatively inexpensive. They can be modelled by a linear time invariant (LTI) system which is in general characterized by its impulse response or step response.

For the measurement of the time response of ultrafast sampling oscilloscopes it is necessary to employ even faster measurement methods. Optoelectronic sampling techniques are well suited for this purpose. Based on such techniques the rise time of the step response of ultrafast sampling oscilloscopes is routinely measured [1,2,3]. Recently, much effort has been spent to extend this single parameter characterization to the measurement of complete time or frequency domain responses, which is referred to as waveform metrology. NIST has developed a combined electric/optoelectronic approach in which the complex transfer function of a sampling oscilloscope is measured up to 110 GHz [4]. In that analysis the uncertainty is propagated using covariance matrices and linearized propagation functions [5]. PTB has just presented a technique in which the step response of a sampling oscilloscope is fully characterized in a 100 ps time window [6]. These results can be used to calculate the oscilloscope's input signal from its output signal, significantly reducing dynamic errors.

In this paper we discuss the significance of correlations in the estimate of the impulse response of sampling oscilloscopes for a subsequent use in the estimation of the input signal. We perform proof-of-principle calculations and show that correlations can have a significant effect. We conclude that correlations should be considered when an estimate of the impulse response is employed for the analysis of oscilloscope measurements.

2. MEASUREMENT UNCERTAINTY

The information provided by a measurement can be seen complete only if it is stated together with its accuracy. In terms of metrology – the science of measurement – this accuracy parameter is the measurement uncertainty. An international guideline for uncertainty evaluation is the *Guide to the Expression of Uncertainty in Measurement* (GUM) [7]. A key feature of the GUM is the propagation of uncertainty which describes how uncertainties associated with the available estimates of all influencing quantities affect the uncertainty of the resulting estimate of the measurand. Assuming a model for the measurand with all influencing input quantities incorporated, this propagation can be computed by linearization as explained in [7]. Although the GUM does not state uncertainty evaluation for dynamic measurements explicitly, the guidelines can be applied for time-domain analysis, [8].

A recently published supplement to that guideline, GUM S1 [9], which we adopt in this paper, replaces the propagation of uncertainty by a propagation of (degree-of-belief) probability density functions (PDFs). Given the model equation of the measurand and the (joint) PDF associated with all input quantities, the PDF assigned to the measurand is then obtained according to the rules of probability theory and it also incorporates possible non-linearities in the evaluation process. The numerical calculation of this PDF can be easily done by a Monte-Carlo procedure [9].

When the information on the input quantities is independent, the joint PDF on all input quantities factorizes into the product of single PDFs assigned to each input quantity. In this case, the estimates $\hat{q}_i, \hat{q}_j$, say, of the different input quantities q_i, q_j are uncorrelated, i.e.,

$$\rho(\hat{q}_i, \hat{q}_j) = u(\hat{q}_i, \hat{q}_j) / u(\hat{q}_i)u(\hat{q}_j) = 0 . \qquad (1)$$

However, this is not always the case and presence of a correlation needs to be accounted for in order to derive a reliable uncertainty for the estimate of the measurand. More precisely, we will show that when an estimate $\hat{h}(t)$ of the impulse response $h(t)$ is used to estimate the oscilloscope's input signal, the correlation $\rho(\hat{h}(t_i), \hat{h}(t_j))$ at different times can play an important role for the size of the resulting uncertainty. Hence, such correlation needs to be determined when the impulse response is estimated from measurements.

3. BASICS OF OSCILLOSCOPE CALIBRATIONS

In this section we introduce some well-known basics required for the understanding of the uncertainty evaluation presented in section 4. The impulse response $h(t)$ of a system corresponds to its output for a unit impulse input signal and fully characterizes the time-domain behaviour of the system. For a general input signal $x(t)$ the corresponding output signal $y(t)$ results from the convolution of the input signal with the impulse response

$$y(t) = (x * h)(t) \quad . \tag{2}$$

An integration of the impulse response results in the step response. The rise time of a system is usually defined as the difference of the 90 % and 10 % quantile of the normalized step response [1]. Early characterization techniques for ultrafast sampling oscilloscopes focused on this quantity as a single parameter characterization that expresses the speed of the oscilloscope. Recently, new analysis techniques have been developed which focus on the measurement of the whole step response (or transfer function) and not just on a single parameter [4,6]. Such measurements are referred to as waveform metrology.

We have performed some simple model calculations to illustrate the importance of waveform metrology and show that such measurements considerably improve the characterization of sampling oscilloscopes as compared to single parameter characterizations. We focus on the impulse response, but of course the same effects are obtained if one considers the step response or the transfer function.

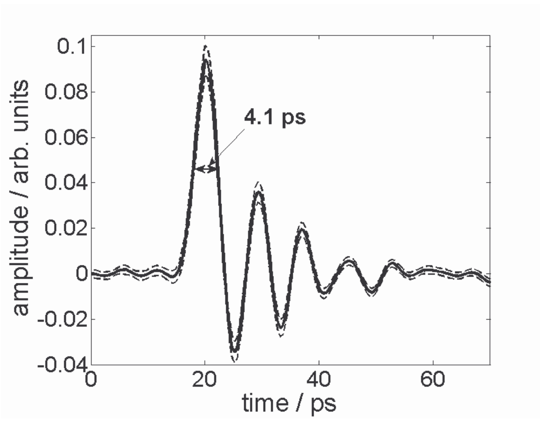

Fig. 1 Impulse response (solid line) with associated standard uncertainty (dashed lines) of a nominal 70 GHz sampling oscilloscope [6].

In figure 1 the measured impulse response of a nominal 70 GHz sampling oscilloscope is shown together with its uncertainty. The data were taken from reference [6].

For a Gaussian pulse with a full-width-at-half-maximum (FWHM) of 4.4 ps as a model input signal (see solid line in the upper part of figure 2) we calculated the output of the oscilloscope using the best estimate of the impulse response according to equation (2). The resulting output signal is shown as dashed line in the upper part of figure 2. To realize the single parameter characterization we took a Gaussian impulse response with the same FWHM as the measured impulse response. An estimate of the oscilloscope's input signal is then obtained by deconvolution of the output signal with the Gaussian impulse response. This estimate is shown as dotted line in the upper part of figure 2. The estimate shows significant dynamic errors, cf. lower part of figure 2. The maximum absolute value of the difference is larger than 30 % of the maximum input signal amplitude, which clearly visualizes the need to perform full waveform metrology instead of a single parameter characterization.

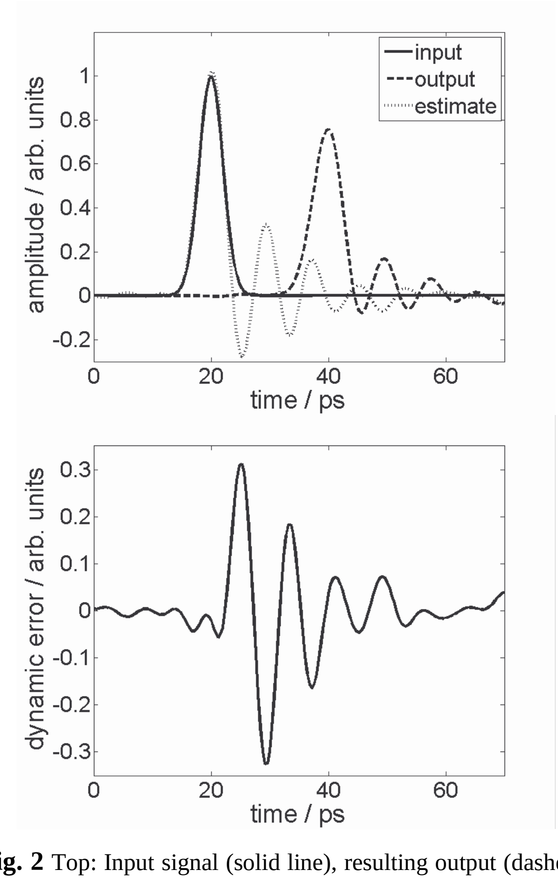

Fig. 2 Top: Input signal (solid line), resulting output (dashed line) and estimated input (dotted line) obtained from deconvolution of the output signal with Gaussian impulse response. Bottom: Relative error of estimation if only a single parameter characterization is applied.

are done in the discrete time and frequency domain, and equations (3,4) are realized by the discrete Fourier transform (DFT) and its inverse.

4. INPUT ESTIMATION AND UNCERTAINTY EVALUATION

We now focus on the uncertainty evaluation of an estimate of the oscilloscope's input signal when it is determined from its output signal and the impulse response. This calculation extends the referenced methods [3-6] in that we consider the estimation of an input signal from the oscilloscope output signal including an uncertainty evaluation according to GUM S1.

In a first step we perform the calculation without considering correlations in the estimate of the impulse response. In a second step we demonstrate that correlations can lead to significant changes in the resulting uncertainties.

The model equation for our calculations reads

$$x(t) = F^{-1}\left\{\frac{Y(\omega)}{H(\omega)}\right\}, \tag{3}$$

where

$$\begin{aligned} H(\omega) &= F\{h(t)\} \\ Y(\omega) &= F\{y(t)\} \end{aligned}, \tag{4}$$

and F and F^{-1} denote Fourier and inverse Fourier transform. The input quantities are the oscilloscope's output signal $y(t)$ and its impulse response $h(t)$.

For our calculations we use the measured impulse response and its associated uncertainty plotted in figure 1. As noted above the data belongs to a calibrated sampling oscilloscope with a nominal bandwidth of 70 GHz [6]. We take as input signal a Gaussian pulse with FWHM of 6 ps and compute the corresponding output signal $y(t)$ by multiplication with $H(\omega)$ in the frequency domain. For the application of equation (2) it is necessary that $|H(\omega)|$ is bounded from below at least in the frequency region which essentially contains the support of $Y(\omega)$. Thus, we here constrain our calculations to the frequency region from zero to 200 GHz where $|H(\omega)|$ is known to be bounded from below and beyond which $Y(\omega)$ is sufficiently close to zero, see figure 3.

For ease of presentation we here assume that $y(t)$ is known exactly, i.e., no noise has corrupted the oscilloscope's output signal. With respect to $h(t)$, we assume an estimate $\hat{h}(t)$ and an associated uncertainty $u(\hat{h}(t))$ is available for each time instant t.

For the evaluation of measurement uncertainty according to GUM S1 a joint PDF is assigned to the discretized impulse response and propagated through the discretized model equation (3) to a PDF for the discrete input signal estimate. To this end, samples from the PDF of the impulse response are drawn repeatedly and propagated through the (discretized) model (3,4), thereby providing samples from the desired PDF of the input signal estimate. All calculations

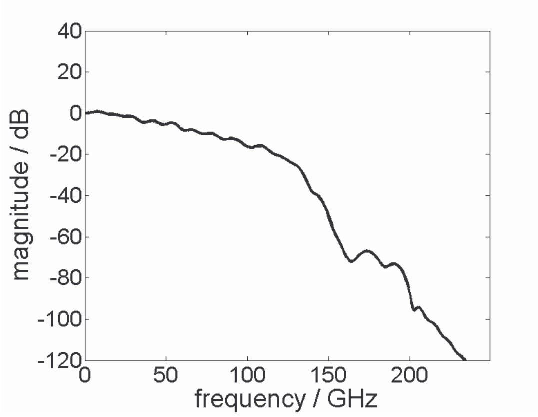

Fig. 3 Normalized magnitude spectrum of the oscilloscope's output signal

Firstly, we assume that no correlation between different times for the estimates of the impulse response is present. Hence, realizations of the impulse response for the Monte-Carlo procedure are drawn by equation (5) where $t_1,\ldots,t_n$ denote the considered discrete times and $N(\hat{\mathbf{h}},\mathbf{U}_{\hat{\mathbf{h}}})$ denotes a multivariate Gaussian distribution with expectation $\hat{\mathbf{h}}$ and diagonal covariance matrix $\mathbf{U}_{\hat{\mathbf{h}}}$.

$$\begin{aligned} \mathbf{h}_k &\sim N\left(\hat{\mathbf{h}},\mathbf{U}_{\hat{\mathbf{h}}}\right) \qquad k = 1,\ldots,M \\ \hat{\mathbf{h}} &= \left(\hat{h}(t_1),\ldots,\hat{h}(t_n)\right)^T \\ \mathbf{U}_{\hat{\mathbf{h}}} &= \begin{pmatrix} u^2(\hat{h}(t_1)) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & u^2(\hat{h}(t_n)) \end{pmatrix} \end{aligned} \tag{5}$$

For each realization of $\mathbf{h}_k$ a corresponding input signal x_k is calculated according to the discretized equation (3,4). The computation of a best estimate for the input signal as the mean and its associated squared uncertainty as the variance results in:

$$\begin{aligned} \hat{x}(t_i) &= \text{mean}_{k=1,\ldots,M}\left\{\left(x_k(t_i)\right)\right\} \\ u^2(\hat{x}(t_i)) &= \text{var}_{k=1,\ldots M}\left\{\left(x_k(t_i)\right)\right\} \end{aligned}. \tag{6}$$

Figure 4 shows the resulting estimate and its associated uncertainty using $M = 10^4$ Monte-Carlo runs. As expected, the estimate $\hat{x}(t)$ coincides with the simulation input signal $x(t)$. Next, we extend the above analysis and consider correlations in the estimate of the impulse response. We assume that the uncertainty matrix $\mathbf{U}_{\hat{\mathbf{h}}}$ has the following artificial correlation structure

$$\mathbf{Corr} = \mathbf{I}(1-\rho) + \rho \cdot \mathbf{1}\mathbf{1}^T \tag{7}$$

where **I** denotes the identity matrix of dimension *n* and **1** an *n*-dimensional vector with all elements equal to 1. Thus, the covariance matrix $\mathbf{U}_{\hat{\mathbf{h}}}$ is computed based on the matrix equivalent of equation (1) which results in

$$(\mathbf{U}_{\hat{\mathbf{h}}})_{ij} = u(\hat{h}(t_i))u(\hat{h}(t_j))(\delta_{ij}(1-\rho)+\rho) \qquad (8)$$

with δ_{ij} being the Kronecker delta. The parameter $\rho \in [-1,1]$ equals the correlation between the estimates for the impulse response at sample times t_i and t_j, $i \neq j$, see section 2. Furthermore, the diagonal elements of $\mathbf{U}_{\hat{\mathbf{h}}}$ are the same as before, while the off-diagonal entries are determined by the condition (7) and controlled by the choice of ρ. Note that for $\rho = 0$ no correlation is present, and the same results as above are obtained.

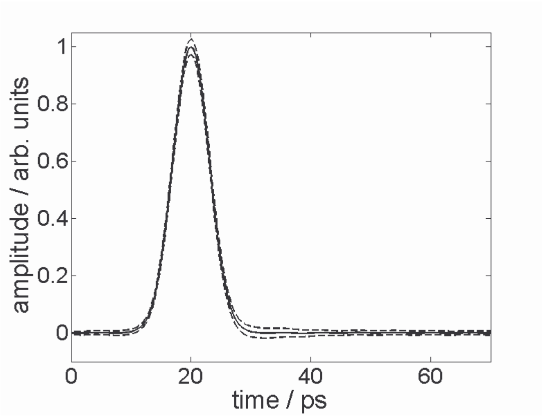

Fig. 4 Estimate of input signal and associated uncertainty.

The calculation scheme for the GUM S1 uncertainty evaluation is the same as for the uncorrelated case, i.e., samples for the impulse response are likewise drawn according to equation (5). Note, however, that now the covariance matrix of the Gaussian distribution in equation (5) is no longer diagonal but replaced by the corresponding non-diagonal uncertainty matrix $\mathbf{U}_{\hat{\mathbf{h}}}$. The resulting estimates are similar to that shown in figure 4, but their uncertainties show a significant dependence on the correlation, cf. figure 5.

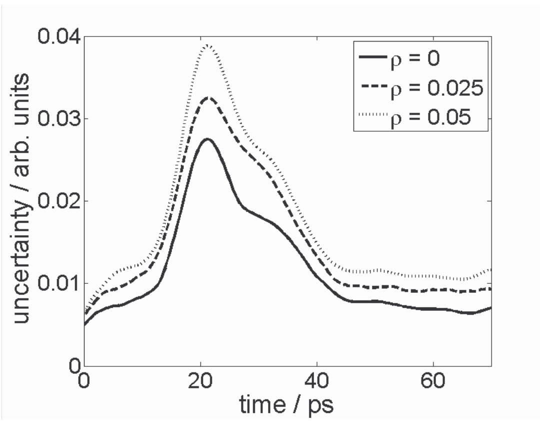

Fig. 5 Result of GUM S1 uncertainty evaluation of the input signal for three different correlation patterns.

For increasing correlation the resulting uncertainty also grows. Note that the shape of the uncertainty does not change, but only its absolute value. This results from the correlation structure of the impulse response which induces, especially in the low-frequency region, an increase of the uncertainty of the transfer function $H(\omega)$ for increasing correlation parameter ρ. Figure 5 shows that already small correlations in the impulse response may significantly influence the uncertainty of the input signal estimate.

5. CONCLUSION

We have demonstrated the significance of correlations in a measured impulse response for uncertainty evaluations of ultrafast sampling oscilloscope measurements. Our proof-of-principle simulations show that such correlations affect the uncertainty evaluation of the oscilloscope's input signal which is calculated using the output signal of the oscilloscope and its impulse response. We hence conclude that correlations need to be considered for a full characterization of the time response of ultrafast sampling oscilloscopes.

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